The Online Math Open Fall Contest
October 25 – November 5, 2019
Acknowledgments

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Contest Information

These rules supersede any rules found elsewhere about the OMO. Please send any further questions directly to the OMO Team at OnlineMathOpenTeam@gmail.com.

Team Registration and Eligibility

Students may compete in teams of up to four people, but no student may belong to more than one team. Participants must not have graduated from high school (or the equivalent secondary school institution in other countries). Teams need not remain the same between the Fall and Spring contests, and students are permitted to participate in whichever contests they like. Only one member on each team needs to register an account on the website. Please check the website, http://internetolympiad.org/pages/14-omo_info, for registration instructions.

Note: when we say “up to four”, we really do mean “up to”! Because the time limit is so long, partial teams are not significantly disadvantaged, and we welcome their participation.

Contest Format and Rules

The 2019 Fall Contest will consist of 30 problems; the answer to each problem will be an integer between 0 and $2^{31} - 1 = 2147483647$ inclusive. The contest window will be

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from 7PM ET on the start day to **7PM ET on the end day**. There is no time limit other than the contest window.

1. Four-function calculators (calculators which can perform only the four basic arithmetic operations) are permitted on the Online Math Open. **Any other computational aids, including scientific calculators, graphing calculators, or computer programs, are prohibited.** All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.

2. Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids, such as Geogebra and graphing calculators, are not allowed. Print and electronic publications are also not allowed.

3. Members of different teams may not communicate with each other about the contest while the contest is running.

4. Anything that may be considered offensive or inappropriate is not allowed as a team name.

5. Team member names must be the actual team members’ names. In particular, names of other people (including celebrities) or phrases that do not represent actual people are not allowed. This includes names of mathematical objects and multiple names concatenated into one name entry.

6. At our discretion, we may rename teams or team members that do not satisfy these rules.

7. Your score is the number of questions answered correctly; that is, every problem is worth one point. Ties will be broken based on the “hardest” problem that a team answered correctly. Remaining ties will be broken by the second hardest problem solved, and so on. (Problem $m$ is harder than problem $n$ if fewer teams solve problem $m$ OR if the number of solves is equal and $m > n$.)

8. Participation in the Online Math Open is free.

Clarifications and Results

Clarifications will be posted as they are answered. For the most recent contests, they will be posted at http://internetolympiad.org/pages/n/omo_problems. If you have a question about problem wording, please email OnlineMathOpenTeam@gmail.com with “Clarification” in the subject. We have the right to deny clarification requests that we feel we cannot answer.
Contest Information

After the contest is over, we will release the answers to the problems within the next day. Please do not discuss the test until answers are released. If you have a protest about an answer, you may send an email to OnlineMathOpenTeam@gmail.com (include “Protest” in the subject). Results will be released in the following weeks. (Results will be counted only for teams that submit answers at least once. Teams that only register an account will not be listed in the final rankings.)
1. Compute the sum of all positive integers \( n \) such that the median of the \( n \) smallest prime numbers is \( n \).

2. Let \( A, B, C, \) and \( P \) be points in the plane such that no three of them are collinear. Suppose that the areas of triangles \( BPC, CPA, \) and \( APB \) are 13, 14, and 15, respectively. Compute the sum of all possible values for the area of triangle \( ABC \).

3. Let \( k \) be a positive real number. Suppose that the set of real numbers \( x \) such that \( x^2 + k|x| \leq 2019 \) is an interval of length 6. Compute \( k \).

4. Maryssa, Stephen, and Cynthia played a game. Each of them independently privately chose one of Rock, Paper, and Scissors at random, with all three choices being equally likely. Given that at least one of them chose Rock and at most one of them chose Paper, the probability that exactly one of them chose Scissors can be expressed as \( \frac{m}{n} \) for relatively prime positive integers \( m \) and \( n \). Compute \( 100m + n \).

5. Compute the number of ordered pairs \((m, n)\) of positive integers that satisfy the equation \( \text{lcm}(m, n) + \gcd(m, n) = m + n + 30 \).

6. An ant starts at the origin of the Cartesian coordinate plane. Each minute it moves randomly one unit in one of the directions up, down, left, or right, with all four directions being equally likely; its direction each minute is independent of its direction in any previous minutes. It stops when it reaches a point \((x, y)\) such that \( |x| + |y| = 3 \). The expected number of moves it makes before stopping can be expressed as \( \frac{m}{n} \) for relatively prime positive integers \( m \) and \( n \). Compute \( 100m + n \).

7. At a concert 10 singers will perform. For each singer \( x \), either there is a singer \( y \) such that \( x \) wishes to perform right after \( y \), or \( x \) has no preferences at all. Suppose that there are \( n \) ways to order the singers such that no singer has an unsatisfied preference, and let \( p \) be the product of all possible nonzero values of \( n \). Compute the largest nonegative integer \( k \) such that \( 2^k \) divides \( p \).

8. There are three eight-digit positive integers which are equal to the sum of the eighth powers of their digits. Given that two of the numbers are 24678051 and 88593477, compute the third number.

9. Convex equiangular hexagon \( ABCDEF \) has \( AB = CD = EF = 1 \) and \( BC = DE = FA = 4 \). Congruent and pairwise externally tangent circles \( \gamma_1, \gamma_2, \) and \( \gamma_3 \) are drawn such that \( \gamma_1 \) is tangent to side \( AB \) and side \( BC \), \( \gamma_2 \) is tangent to side \( CD \) and side \( DE \), and \( \gamma_3 \) is tangent to side \( EF \) and side \( FA \). Then the area of \( \gamma_1 \) can be expressed as \( \frac{m}{n} \) for relatively prime positive integers \( m \) and \( n \). Compute \( 100m + n \).

10. Let \( k \) be a positive integer. Marco and Vera play a game on an infinite grid of square cells. At the beginning, only one cell is black and the rest are white.

A turn in this game consists of the following. Marco moves first, and for every move he must choose a cell which is black and which has more than two white neighbors. (Two cells are neighbors if they share an edge, so every cell has exactly four neighbors.) His move consists of making the chosen black cell white and turning all of its neighbors black if they are not already. Vera then performs the following action exactly \( k \) times: she chooses two cells that are neighbors to each other and swaps their colors (she is allowed to swap the colors of two white or of two black cells, though doing so has no effect).

This, in totality, is a single turn. If Vera leaves the board so that Marco cannot choose a cell that is black and has more than two white neighbors, then Vera wins; otherwise, another turn occurs.

Let \( m \) be the minimal \( k \) value such that Vera can guarantee that she wins no matter what Marco does. For \( k = m \), let \( t \) be the smallest positive integer such that Vera can guarantee, no matter what Marco does, that she wins after at most \( t \) turns. Compute \( 100m + t \).

11. Let \( ABC \) be a triangle with incenter \( I \) such that \( AB = 20 \) and \( AC = 19 \). Point \( P \neq A \) lies on line \( AB \) and point \( Q \neq A \) lies on line \( AC \). Suppose that \( IA = IP = IQ \) and that line \( PQ \) passes through the midpoint of side \( BC \). Suppose that \( BC = \frac{m}{n} \) for relatively prime positive integers \( m \) and \( n \). Compute \( 100m + n \).
12. Let $F(n)$ denote the smallest positive integer greater than $n$ whose sum of digits is equal to the sum of the digits of $n$. For example, $F(2019) = 2028$. Compute $F(1) + F(2) + \cdots + F(1000)$.

13. Compute the number of subsets $S$ with at least two elements of $\{2^2, 3^3, \ldots, 216^{16}\}$ such that the product of the elements of $S$ has exactly 216 positive divisors.

14. The sequence of nonnegative integers $F_0, F_1, F_2, \ldots$ is defined recursively as $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all integers $n \geq 0$. Let $d$ be the largest positive integer such that, for all integers $n \geq 0$, $d$ divides $F_{n+2020} - F_n$. Compute the remainder when $d$ is divided by 1001.

15. Let $A, B, C,$ and $D$ be points in the plane with $AB = AC = BC = BD = CD = 36$ and such that $A \neq D$. Point $K$ lies on segment $AC$ such that $AK = 2KC$. Point $M$ lies on segment $AB$, and point $N$ lies on line $AC$, such that $D$, $M$, and $N$ are collinear. Let lines $CM$ and $BN$ intersect at $P$. Then the maximum possible length of segment $KP$ can be expressed in the form $m + \sqrt{n}$ for positive integers $m$ and $n$. Compute $100m + n$.

16. Let $ABC$ be a scalene triangle with inradius $r_A$, $r_B$, and $r_C$ such that

$$20 (r_B^2 r_C^2 + r_C^2 r_A^2 + r_A^2 r_B^2) = 19 (r_{ABC})^2.$$ 

If \[
\frac{\tan A}{2} + \frac{\tan B}{2} + \frac{\tan C}{2} = 2.019,
\]

then the area of $\triangle ABC$ can be expressed as $\frac{mw}{n}$ for relatively prime positive integers $m$ and $n$. Compute $100m + n$.

17. For an ordered pair $(m, n)$ of distinct positive integers, suppose, for some nonempty subset $S$ of $\mathbb{R}$, that a function $f : S \to S$ satisfies the property that $f^n(x) + f^n(y) = x + y$ for all $x, y \in S$. (Here $f^k(z)$ means the result when $f$ is applied $k$ times to $z$; for example, $f^3(z) = f(f(f(z)))$.) Then $f$ is called $(m, n)$-splendid. Furthermore, $f$ is called $(m, n)$-primitive if $f$ is $(m, n)$-splendid and there do not exist positive integers $a \leq m$ and $b \leq n$ with $(a, b) \neq (m, n)$ and $a \neq b$ such that $f$ is also $(a, b)$-splendid. Compute the number of ordered pairs $(m, n)$ of distinct positive integers less than 10000 such that there exists a nonempty subset $S$ of $\mathbb{R}$ such that there exists an $(m, n)$-primitive function $f : S \to S$.

18. Define a modern artwork to be a nonempty finite set of rectangles in the Cartesian coordinate plane with positive areas, pairwise disjoint interiors, and sides parallel to the coordinate axes. For a modern artwork $S$, define its price to be the minimum number of colors with which Sean could paint the interiors of rectangles in $S$ such that every rectangle’s interior is painted in exactly one color and every two distinct touching rectangles have distinct colors, where two rectangles are touching if they share infinitely many points. For a positive integer $n$, let $g(n)$ denote the maximum price of any modern artwork with exactly $n$ rectangles. Compute $g(1) + g(2) + \cdots + g(2019)$.

19. Let $ABC$ be an acute triangle with circumcenter $O$ and orthocenter $H$. Let $E$ be the intersection of $BH$ and $AC$ and let $M$ and $N$ be the midpoints of $HB$ and $HO$, respectively. Let $I$ be the incenter of $AEM$ and $J$ be the intersection of $ME$ and $AI$. If $AO = 20$, $AN = 17$, and $\angle ANM = 90^\circ$, then $\frac{AI}{AJ} = \frac{m}{n}$ for relatively prime positive integers $m$ and $n$. Compute $100m + n$.

20. Define a crossword puzzle to be a $15 \times 15$ grid of squares, each of which is either black or white. In a crossword puzzle, define a word to be a sequence of one or more consecutive white squares in a row or column such that the squares immediately before and after the sequence both are either black or do not exist. (The latter case would occur if an end of a word coincides with an end of a row or column of the grid.) A crossword puzzle is tasty if every word consists of an even number of white squares. Compute the sum of all nonnegative integers $n$ such that there exists a tasty crossword puzzle with exactly $n$ white squares.

21. Let $p$ and $q$ be prime numbers such that $(p - 1)^{q-1} - 1$ is a positive integer that divides $(2q)^{2p} - 1$. Compute the sum of all possible values of $pq$. 


22. For finite sets $A$ and $B$, call a function $f : A \to B$ an \textit{antibijection} if there does not exist a set $S \subseteq A \cap B$ such that $S$ has at least two elements and, for all $s \in S$, there exists exactly one element $s'$ of $S$ such that $f(s') = s$. Let $N$ be the number of antibijections from $\{1, 2, 3, \ldots, 2019\}$ to $\{1, 2, 3, \ldots, 2019\}$. Suppose $N$ is written as the product of a collection of (not necessarily distinct) prime numbers. Compute the sum of the members of this collection. (For example, if it were true that $N = 12 = 2 \times 2 \times 3$, then the answer would be $2 + 2 + 3 = 7$.)

23. Let $v$ and $w$ be real numbers such that, for all real numbers $a$ and $b$, the inequality

$$(2^{a+b} + 8)(3^a + 3^b) \leq v(12^{a-1} + 12^{b-1} - 2^{a+b-1}) + w$$

holds. Compute the smallest possible value of $128v^2 + w^2$.

24. Let $ABC$ be an acute scalene triangle with orthocenter $H$ and circumcenter $O$. Let the line through $A$ tangent to the circumcircle of triangle $AHO$ intersect the circumcircle of triangle $ABC$ at $A$ and $P \neq A$. Let the circumcircles of triangles $AOP$ and $BHP$ intersect at $P$ and $Q \neq P$. Let line $PQ$ intersect segment $BO$ at $X$. Suppose that $BX = 2$, $OX = 1$, and $BC = 5$. Then $AB \cdot AC = \sqrt{k} + m \sqrt{n}$ for positive integers $k$, $m$, and $n$, where neither $k$ nor $n$ is divisible by the square of any integer greater than $1$. Compute $100k + 10m + n$.

25. The sequence $f_0, f_1, \ldots$ of polynomials in $\mathbb{F}_11[x]$ is defined by $f_0(x) = x$ and $f_{n+1}(x) = f_n(x)^{11} - f_n(x)$ for all $n \geq 0$. Compute the remainder when the number of nonconstant monic irreducible divisors of $f_{1000}(x)$ is divided by $1000$.

26. Let $p = 491$ be prime. Let $S$ be the set of ordered $k$-tuples of nonnegative integers that are less than $p$. We say that a function $f : S \to S$ is $k$-\textit{murnie} if, for all $u, v \in S$, $\langle f(u), f(v) \rangle \equiv \langle u, v \rangle \pmod{p}$, where $\langle (a_1, \ldots, a_k), (b_1, \ldots, b_k) \rangle = a_1b_1 + \cdots + a_kb_k$ for any $(a_1, \ldots, a_k), (b_1, \ldots, b_k) \in S$.

Let $m(k)$ be the number of $k$-murnie functions. Compute the remainder when $m(1) + m(2) + m(3) + \cdots + m(p)$ is divided by $488$.

27. A \textit{complex set}, along with its complexity, is defined recursively as the following:

- The set $\mathbb{C}$ of complex numbers is a complex set with complexity $1$.
- Given two complex sets $C_1, C_2$ with complexity $c_1, c_2$ respectively, the set of all functions $f : C_1 \to C_2$ is a complex set denoted $[C_1, C_2]$ with complexity $c_1 + c_2$.

A \textit{complex expression}, along with its evaluation and its complexity, is defined recursively as the following:

- A single complex set $C$ with complexity $c$ is a complex expression with complexity $c$ that evaluates to itself.
- Given two complex expressions $E_1, E_2$ with complexity $c_1, c_2$ that evaluate to $C_1$ and $C_2$ respectively, if $C_1 = [C_2, C]$ for some complex set $C$, then $(E_1, E_2)$ is a complex expression with complexity $c_1 + c_2$ that evaluates to $C$.

For a positive integer $n$, let $a_n$ be the number of complex expressions with complexity $n$ that evaluate to $\mathbb{C}$. Let $x$ be a positive real number. Suppose that

$$a_1 + a_2 x + a_3 x^2 + \cdots = \frac{7}{4}.$$ 

Then $x = \frac{k \sqrt{m}}{n}$, where $k, m, n$ are positive integers such that $m$ is not divisible by the square of any integer greater than $1$, and $k$ and $n$ are relatively prime. Compute $100k + 10m + n$.

28. Let $S$ be the set of integers modulo $2020$. Suppose that $a_1, a_2, \ldots, a_{2020}, b_1, b_2, \ldots, b_{2020}, c$ are arbitrary elements of $S$. For any $x_1, x_2, \ldots, x_{2020} \in S$, define $f(x_1, x_2, \ldots, x_{2020})$ to be the $2020$-tuple whose $i$th coordinate is $x_{i-2} + a_i x_{2019} + b_i x_{2020} + c x_i$, where we set $x_{-1} = x_0 = 0$. Let $m$ be the smallest positive
integer such that, for some values of \(a_1, a_2, \ldots, a_{2020}, b_1, b_2, \ldots, b_{2020}, c\), we have, for all \(x_1, x_2, \ldots, x_{2020} \in S\), that \(f^m(x_1, x_2, \ldots, x_{2020}) = (0, 0, \ldots, 0)\). For this value of \(m\), there are exactly \(n\) choices of the tuple \((a_1, a_2, \ldots, a_{2020}, b_1, b_2, \ldots, b_{2020}, c)\) such that, for all \(x_1, x_2, \ldots, x_{2020} \in S\), \(f^m(x_1, x_2, \ldots, x_{2020}) = (0, 0, \ldots, 0)\). Compute \(100m + n\).

29. Let \(ABC\) be a triangle. The line through \(A\) tangent to the circumcircle of \(ABC\) intersects line \(BC\) at point \(W\). Points \(X, Y \neq A\) lie on lines \(AC\) and \(AB\), respectively, such that \(WA = WX = WY\). Point \(X_1\) lies on line \(AB\) such that \(\angle AXX_1 = 90^\circ\), and point \(X_2\) lies on line \(AC\) such that \(\angle AX_1X_2 = 90^\circ\). Point \(Y_1\) lies on line \(AC\) such that \(\angle AY_1Y = 90^\circ\), and point \(Y_2\) lies on line \(AB\) such that \(\angle AY_1Y_2 = 90^\circ\). Let lines \(AW\) and \(XY\) intersect at point \(Z\), and let point \(P\) be the foot of the perpendicular from \(A\) to line \(X_2Y_2\). Let line \(ZP\) intersect line \(BC\) at \(U\) and the perpendicular bisector of segment \(BC\) at \(V\). Suppose that \(C\) lies between \(B\) and \(U\). Let \(x\) be a positive real number. Suppose that \(AB = x + 1\), \(AC = 3\), \(AV = x\), and \(\frac{BC}{CD} = x\). Then \(x = \frac{\sqrt{3} - m}{n}\) for positive integers \(k, m,\) and \(n\) such that \(k\) is not divisible by the square of any integer greater than 1. Compute \(100k + 10m + n\).

30. For a positive integer \(n\), we say an \(n\)-transposition is a bijection \(\sigma : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}\) such that there exist exactly two elements \(i\) of \(\{1, 2, \ldots, n\}\) such that \(\sigma(i) \neq i\).

Fix some four pairwise distinct \(n\)-transpositions \(\sigma_1, \sigma_2, \sigma_3, \sigma_4\). Let \(q\) be any prime, and let \(\mathbb{F}_q\) be the integers modulo \(q\). Consider all functions \(f : (\mathbb{F}_q^n) \rightarrow \mathbb{F}_q\) that satisfy, for all integers \(i\) with \(1 \leq i \leq n\) and all \(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y, z \in \mathbb{F}_q^n\),

\[
f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) + f(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, y + z, x_{i+1}, \ldots, x_n),
\]

and that satisfy, for all \(x_1, \ldots, x_n \in \mathbb{F}_q^n\) and all \(\sigma \in \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}\),

\[
f(x_1, \ldots, x_n) = -f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).
\]

(Note that the equalities in the previous sentence are in \(\mathbb{F}_q\). Note that, for any \(a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{F}_q\), we have \((a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)\), where \(a_1 + b_1, \ldots, a_n + b_n \in \mathbb{F}_q\).)

For a given tuple \((x_1, \ldots, x_n) \in (\mathbb{F}_q^n)^n\), let \(g(x_1, \ldots, x_n)\) be the number of different values of \(f(x_1, \ldots, x_n)\) over all possible functions \(f\) satisfying the above conditions.

Pick \((x_1, \ldots, x_n) \in (\mathbb{F}_q^n)^n\) uniformly at random, and let \(\varepsilon(q, \sigma_1, \sigma_2, \sigma_3, \sigma_4)\) be the expected value of \(g(x_1, \ldots, x_n)\). Finally, let

\[
\kappa(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = -\lim_{q \to \infty} \log_q \left( -\ln \left( \frac{\varepsilon(q, \sigma_1, \sigma_2, \sigma_3, \sigma_4) - 1}{q - 1} \right) \right).
\]

Pick four pairwise distinct \(n\)-transpositions \(\sigma_1, \sigma_2, \sigma_3, \sigma_4\) uniformly at random from the set of all \(n\)-transpositions. Let \(\pi(n)\) denote the expected value of \(\kappa(\sigma_1, \ldots, \sigma_4)\). Suppose that \(p(x)\) and \(q(x)\) are polynomials with real coefficients such that \(q(-3) \neq 0\) and such that \(\pi(n) = \frac{p(n)}{q(n)}\) for infinitely many positive integers \(n\). Compute \(\frac{p(-3)}{q(-3)}\).