Problem 1. Let $ABC$ be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let $D$ be the point inside triangle $ABC$ with the property that $BD \perp CD$ and $AD \perp BC$. Then the length $AD$ can be expressed in the form $m - \sqrt{n}$, where $m$ and $n$ are positive integers. Find $100m + n$.

Problem 2. In the figure below, how many ways are there to select two squares which do not share an edge?

Problem 3. Let $S = \{1, 2, \ldots, 2014\}$. Suppose that

$$\sum_{T \subseteq S} i^{\left|T\right|} = p + qi$$

where $p$ and $q$ are integers, $i = \sqrt{-1}$, and the summation runs over all $2^{2014}$ subsets of $S$. Find the remainder when $|p| + |q|$ is divided by 1000. (Here $|X|$ denotes the number of elements in a set $X$.)

Problem 4. Points $A$, $B$, $C$, and $D$ lie on a circle such that chords $AC$ and $BD$ intersect at a point $E$ inside the circle. Suppose that $\angle ADE = \angle CBE = 75^\circ$, $BE = 4$, and $DE = 8$. The value of $AB^2$ can be written in the form $a + b\sqrt{c}$ for positive integers $a$, $b$, and $c$ such that $c$ is not divisible by the square of any prime. Find $a + b + c$.

Problem 5. Let $r$, $s$, $t$ be the roots of the polynomial $x^3 + 2x^2 + x - 7$. Then

$$\left(1 + \frac{1}{(r + 2)^2}\right)\left(1 + \frac{1}{(s + 2)^2}\right)\left(1 + \frac{1}{(t + 2)^2}\right) = \frac{m}{n}$$

for relatively prime positive integers $m$ and $n$. Compute $100m + n$.

Problem 6. For all positive integers $k$, define $f(k) = k^2 + k + 1$. Compute the largest positive integer $n$ such that

$$2015f(1^2)f(2^2)\cdots f(n^2) \geq \left(f(1)f(2)\cdots f(n)\right)^2.$$

Problem 7. Find the sum of the prime factors of 67208001, given that 23 is one.

Problem 8. For positive integers $a$, $b$, and $c$, define

$$f(a, b, c) = \frac{abc}{\gcd(a, b, c) \cdot \text{lcm}(a, b, c)}.$$

We say that a positive integer $n$ is $f@$ if there exist pairwise distinct positive integers $x, y, z \leq 60$ that satisfy $f(x, y, z) = n$. How many $f@$ integers are there?

Time limit: 40 minutes
Maximum score is 56 points