NIMO Winter Contest 2014

1. This is an eight-problem exam. Your team of up to four students will be allotted 60 minutes to complete the exam.

2. **No aids other that scratch paper, graph paper, rulers, compasses, and protractors are permitted**. In particular, calculators, slide rules, or other computational aids are not allowed. Use of computers is permitted only for communication among team members and for typing and/or submitting solutions.

3. You must show work and justify your reasoning to receive credit. Answers without justification will receive little or no credit. Conversely, a well-written solution with a few minor errors may receive near-full or full credit. Each problem will be given a score from 0 to 7 based on correctness, completeness, method of attack, and clarity of reasoning; the scoring will be similar to that of the International Mathematical Olympiad.

4. Solutions may be either typed or handwritten. There is a 16MB file limit for uploaded solutions, so we encourage typing for uploaded solutions.

   - If you choose to type your solutions, we strongly encourage you to use LaTeX. Note that your solutions must still be uploaded as a PDF should you submit them electronically.
   - If you choose to write your solutions by hand, make sure your solution is legible and dark enough to be processed by a machine - we cannot grade solutions that are unreadable.

   No page should include solutions to more than one problem – even if your solution is less than one page, please start a new page for each problem.

5. You must submit your solutions on or before January 13, 2014. You must submit your solutions by one of the following methods:

   - Mail all of your solutions with cover sheet (details on our website) in order in a single envelope addressed to:

     NIMO 2014 Winter Contest
P.O. Box 11
Fremont, CA 94537

     Solutions must be postmarked on or before January 13, 2014. Note that this address is not the same as least year's, and is likely to change again next year.
   - Upload your solutions in a single PDF file through your team leader's account at our site. You will be allotted up to 30 minutes after the contest to scan and upload your solutions.

6. **Please read all of the rules on the NIMO website before beginning the exam.**

5^{th} National Internet Mathematical Olympiad

www.internetolympiad.org

Winter Contest NIMO 2014

January 1 – January 13

1. Find, with proof, all real numbers x satisfying $x = 2\left(2\left(2\left(2\left(2x-1\right)-1\right)-1\right)-1\right)-1$.

2. Determine, with proof, the smallest positive integer c such that for any positive integer n, the decimal representation of the number $c^n + 2014$ has digits all less than 5.

3. The numbers $1, 2, \dots, 10$ are written on a board. Every minute, one can select three numbers a, b, c on the board, erase them, and write $\sqrt{a^2+b^2+c^2}$ in their place. This process continues until no more numbers can be erased. What is the largest possible number that can remain on the board at this point?

4. Prove that there exist integers a, b, c with $1 \le a < b < c \le 25$ and
$$S\left(a^6+2014\right) = S\left(b^6+2014\right) = S\left(c^6+2014\right)$$
where $S(n)$ denotes the sum of the decimal digits of n.

5. Let ABC be an acute triangle with orthocenter H and let M be the midpoint of $\overline{BC}$. (The *orthocenter* is the point at the intersection of the three altitudes.) Denote by ω_B the circle passing through B, H, and M, and denote by ω_C the circle passing through C, H, and M. Lines AB and AC meet ω_B and ω_C again at P and Q, respectively. Rays PH and QH meet ω_C and ω_B again at R and S, respectively. Show that $\triangle BRS$ and $\triangle CRS$ have the same area.

6. Let $\varphi(k)$ denote the numbers of positive integers less than or equal to k and relatively prime to k. Prove that for some positive integer n,
$$\varphi(2n-1) + \varphi(2n+1) < \frac{1}{1000}\varphi(2n).$$

7. Let ABC be a triangle and let Q be a point such that $\overline{AB} \perp \overline{QB}$ and $\overline{AC} \perp \overline{QC}$. A circle with center I is inscribed in $\triangle ABC$, and is tangent to $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ at points D, E, and F, respectively. If ray QI intersects $\overline{EF}$ at P, prove that $\overline{DP} \perp \overline{EF}$.

8. Define the function $\xi : \mathbb{Z}^2 \to \mathbb{Z}$ by $\xi(n,k) = 1$ when $n \le k$ and $\xi(n,k) = -1$ when $n > k$, and construct the polynomial
$$P(x_1, \dots, x_{1000}) = \prod_{n=1}^{1000} \left(\sum_{k=1}^{1000} \xi(n,k) x_k \right).$$
   (a) Determine the coefficient of $x_1 x_2 \dots x_{1000}$ in P.
   (b) Show that if $x_1, x_2, \dots, x_{1000} \in \{-1, 1\}$ then $P(x_1, x_2, \dots, x_{1000}) = 0$.

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