The Online Math Open Fall Contest
October 17 - 28, 2014

Acknowledgements

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Contest Information

These rules supersede any rules found elsewhere about the OMO. Please send any further questions directly to the OMO Team at `OnlineMathOpenTeam@gmail.com`.

Team Registration and Eligibility

Students may compete in teams of up to four people, but no student can belong to more than one team. Participants must not have graduated from high school (or the equivalent secondary school institution in other countries). Teams need not remain the same between the Fall and Spring contests, and students are permitted to participate in whichever contests they like.

Only one member on each team needs to register an account on the website. Please check the website, `http://internetolympiad.org/pages/14-omo_info`, for registration instructions.

Note: when we say "up to four", we really do mean "up to"! Because the time limit is so long, partial teams are not significantly disadvantaged, and we welcome their participation.

Contest Format and Rules

The 2014 Fall Contest will consist of 30 problems; the answer to each problem will be an integer between 0 and $2^{31} - 1 = 2147483647$ inclusive. The contest window will be

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from 7PM ET on the start day to **7PM ET on the end day**. There is no time limit other than the contest window.

1. Four-function calculators (calculators which can perform only the four basic arithmetic operations) are permitted on the Online Math Open. **Any other computational aids, including scientific calculators, graphing calculators, or computer programs is prohibited.** All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.

2. Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids, such as Geogebra and graphing calculators, are not allowed. Print and electronic publications are also not allowed.

3. Members of different teams cannot communicate with each other about the contest while the contest is running.

4. Your score is the number of questions answered correctly; that is, every problem is worth one point. Ties will be broken based on the "hardest" problem that a team answered correctly. Remaining ties will be broken by the second hardest problem solved, and so on. (Problem m is harder than problem n if fewer teams solve problem m OR if the number of solves is equal and $m > n$.)

5. *Participation in the Online Math Open is free.*

Clarifications and Results

Clarifications will be posted as they are answered. For the most recent contests, they will be posted at `http://internetolympiad.org/pages/n/omo_problems`. If you have a question about problem wording, please email `OnlineMathOpenTeam@gmail.com` with "Clarification" in the subject. We have the right to deny clarification requests that we feel we cannot answer.

After the contest is over, we will release the answers to the problems within the next day. Please do not discuss the test until answers are released. If you have a protest about an answer, you may send an email to `OnlineMathOpenTeam@gmail.com`. (Include "Protest" in the subject). Results will be released in the following weeks. (Results will be counted only for teams that submit answers at least once. Teams that only register an account will not be listed in the final rankings.)

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1. Carl has a rectangle whose side lengths are positive integers. This rectangle has the property that when he increases the width by 1 unit and decreases the length by 1 unit, the area increases by x square units. What is the smallest possible positive value of x?

2. Suppose (a_n), (b_n), (c_n) are arithmetic progressions. Given that $a_1 + b_1 + c_1 = 0$ and $a_2 + b_2 + c_2 = 1$, compute $a_{2014} + b_{2014} + c_{2014}$.

3. Let $B = (20, 14)$ and $C = (18, 0)$ be two points in the plane. For every line ℓ passing through B, we color red the foot of the perpendicular from C to ℓ. The set of red points enclose a bounded region of area $\mathcal{A}$. Find $\lfloor \mathcal{A} \rfloor$ (that is, find the greatest integer not exceeding $\mathcal{A}$).

4. A crazy physicist has discovered a new particle called an emon. He starts with two emons in the plane, situated a distance 1 from each other. He also has a crazy machine which can take any two emons and create a third one in the plane such that the three emons lie at the vertices of an equilateral triangle. After he has five total emons, let P be the product of the $\binom{5}{2} = 10$ distances between the 10 pairs of emons. Find the greatest possible value of P^2.

5. A crazy physicist has discovered a new particle called an omon. He has a machine, which takes two omons of mass a and b and entangles them; this process destroys the omon with mass a, preserves the one with mass b, and creates a new omon whose mass is $\frac{1}{2}(a + b)$. The physicist can then repeat the process with the two resulting omons, choosing which omon to destroy at every step. The physicist initially has two omons whose masses are distinct positive integers less than 1000. What is the maximum possible number of times he can use his machine without producing an omon whose mass is not an integer?

6. For an olympiad geometry problem, Tina wants to draw an acute triangle whose angles each measure a multiple of $10°$. She doesn't want her triangle to have any special properties, so none of the angles can measure $30°$ or $60°$, and the triangle should definitely not be isosceles.

   How many different triangles can Tina draw? (Similar triangles are considered the same.)

7. Define the function $f(x, y, z)$ by
$$f(x, y, z) = x^{y^z} - x^{z^y} + y^{z^x} - y^{x^z} + z^{x^y}.$$
   Evaluate $f(1, 2, 3) + f(1, 3, 2) + f(2, 1, 3) + f(2, 3, 1) + f(3, 1, 2) + f(3, 2, 1)$.

8. Let a and b be randomly selected three-digit integers and suppose $a > b$. We say that a is *clearly bigger* than b if each digit of a is larger than the corresponding digit of b. If the probability that a is clearly bigger than b is $\frac{m}{n}$, where m and n are relatively prime integers, compute $m + n$.

9. Let $N = 2014! + 2015! + 2016! + \cdots + 9999!$. How many zeros are at the end of the decimal representation of N?

10. Find the sum of the decimal digits of
$$\left\lfloor \frac{51525354555657\ldots979899}{50} \right\rfloor.$$
    Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.

11. Given a triangle ABC, consider the semicircle with diameter $\overline{EF}$ on $\overline{BC}$ tangent to $\overline{AB}$ and $\overline{AC}$. If $BE = 1$, $EF = 24$, and $FC = 3$, find the perimeter of $\triangle ABC$.

12. Let a, b, c be positive real numbers for which
$$\frac{5}{a} = b + c, \quad \frac{10}{b} = c + a, \quad \text{and} \quad \frac{13}{c} = a + b.$$
    If $a + b + c = \frac{m}{n}$ for relatively prime positive integers m and n, compute $m + n$.

13. Two ducks, Wat and Q, are taking a math test with 1022 other ducklings. The test has 30 questions, and the nth question is worth n points. The ducks work independently on the test. Wat gets the nth problem correct with probability $\frac{1}{n^2}$ while Q gets the nth problem correct with probability $\frac{1}{n+1}$. Unfortunately, the remaining ducklings each answer all 30 questions incorrectly.

    Just before turning in their test, the ducks and ducklings decide to share answers! On any question which Wat and Q have the same answer, the ducklings change their answers to agree with them. After this process, what is the expected value of the sum of all 1024 scores?

14. What is the greatest common factor of 12345678987654321 and 12345654321?

15. Let $\phi = \frac{1+\sqrt{5}}{2}$. A *base-$\phi$ number* $(a_n a_{n-1} \dots a_1 a_0)_\phi$, where $0 \le a_n, a_{n-1}, \dots, a_0 \le 1$ are integers, is defined by
$$(a_n a_{n-1} \dots a_1 a_0)_\phi = a_n \cdot \phi^n + a_{n-1} \cdot \phi^{n-1} + \dots + a_1 \cdot \phi^1 + a_0.$$
    Compute the number of base-ϕ numbers $(b_j b_{j-1} \dots b_1 b_0)_\phi$ which satisfy $b_j \neq 0$ and
$$(b_j b_{j-1} \dots b_1 b_0)_\phi = \underbrace{(100\dots100)_\phi}_{\text{Twenty } 100's}.$$

16. Let $OABC$ be a tetrahedron such that $\angle AOB = \angle BOC = \angle COA = 90^\circ$ and its faces have integral surface areas. If $[OAB] = 20$ and $[OBC] = 14$, find the sum of all possible values of $[OCA][ABC]$. (Here $[\triangle]$ denotes the area of $\triangle$.)

17. Let ABC be a triangle with area 5 and $BC = 10$. Let E and F be the midpoints of sides AC and AB respectively, and let BE and CF intersect at G. Suppose that quadrilateral $AEGF$ can be inscribed in a circle. Determine the value of $AB^2 + AC^2$.

18. We select a real number α uniformly and at random from the interval $(0, 500)$. Define
$$S = \frac{1}{\alpha} \sum_{m=1}^{1000} \sum_{n=m}^{1000} \left\lfloor \frac{m+\alpha}{n} \right\rfloor.$$
    Let p denote the probability that $S \ge 1200$. Compute $1000p$.

19. In triangle ABC, $AB = 3$, $AC = 5$, and $BC = 7$. Let E be the reflection of A over $\overline{BC}$, and let line BE meet the circumcircle of ABC again at D. Let I be the incenter of $\triangle ABD$. Given that $\cos^2 \angle AEI = \frac{m}{n}$, where m and n are relatively prime positive integers, determine $m + n$.

20. Let $n = 2188 = 3^7 + 1$ and let $A_0^{(0)}, A_1^{(0)}, \dots, A_{n-1}^{(0)}$ be the vertices of a regular n-gon (in that order) with center O . For $i = 1, 2, \dots, 7$ and $j = 0, 1, \dots, n-1$, let $A_j^{(i)}$ denote the centroid of the triangle
$$\triangle A_j^{(i-1)} A_{j+3^{7-i}}^{(i-1)} A_{j+2\cdot 3^{7-i}}^{(i-1)}.$$
    Here the subscripts are taken modulo n. If
$$\frac{|OA_{2014}^{(7)}|}{|OA_{2014}^{(0)}|} = \frac{p}{q}$$
    for relatively prime positive integers p and q, find $p + q$.

21. Consider a sequence $x_1, x_2, \cdots x_{12}$ of real numbers such that $x_1 = 1$ and for $n = 1, 2, \dots, 10$ let
$$x_{n+2} = \frac{(x_{n+1}+1)(x_{n+1}-1)}{x_n}.$$
    Suppose $x_n > 0$ for $n = 1, 2, \dots, 11$ and $x_{12} = 0$. Then the value of x_2 can be written as $\frac{\sqrt{a}+\sqrt{b}}{c}$ for positive integers a, b, c with $a > b$ and no square dividing a or b. Find $100a + 10b + c$.

22. Find the smallest positive integer c for which the following statement holds: Let k and n be positive integers. Suppose there exist pairwise distinct subsets $S_1, S_2, \ldots, S_{2k}$ of $\{1, 2, \ldots, n\}$, such that $S_i \cap S_j \neq \varnothing$ and $S_i \cap S_{j+k} \neq \varnothing$ for all $1 \leq i, j \leq k$. Then $1000k \leq c \cdot 2^n$.

23. For a prime q, let $\Phi_q(x) = x^{q-1} + x^{q-2} + \cdots + x + 1$. Find the sum of all primes p such that $3 \leq p \leq 100$ and there exists an odd prime q and a positive integer N satisfying

$$\binom{N}{\Phi_q(p)} \equiv \binom{2\Phi_q(p)}{N} \not\equiv 0 \pmod{p}.$$

24. Let $\mathcal{A} = A_0A_1A_2A_3 \cdots A_{2013}A_{2014}$ be a *regular 2014-simplex*, meaning the 2015 vertices of $\mathcal{A}$ lie in 2014-dimensional Euclidean space and there exists a constant $c > 0$ such that $A_iA_j = c$ for any $0 \leq i < j \leq 2014$. Let $O = (0, 0, 0, \ldots, 0)$, $A_0 = (1, 0, 0, \ldots, 0)$, and suppose A_iO has length 1 for $i = 0, 1, \ldots, 2014$. Set $P = (20, 14, 20, 14, \ldots, 20, 14)$. Find the remainder when

$$PA_0^2 + PA_1^2 + \cdots + PA_{2014}^2$$

is divided by 10^6.

25. Kevin has a set S of 2014 points scattered on an infinitely large planar gameboard. Because he is bored, he asks Ashley to evaluate

$$x = 4f_4 + 6f_6 + 8f_8 + 10f_{10} + \cdots$$

while he evaluates

$$y = 3f_3 + 5f_5 + 7f_7 + 9f_9 + \cdots,$$

where f_k denotes the number of convex k-gons whose vertices lie in S but none of whose interior points lie in S. However, since Kevin wishes to one-up everything that Ashley does, he secretly positions the points so that $y - x$ is as large as possible, but in order to avoid suspicion, he makes sure no three points lie on a single line. Find $|y - x|$.

26. Let ABC be a triangle with $AB = 26$, $AC = 28$, $BC = 30$. Let X, Y, Z be the midpoints of arcs BC, CA, AB (not containing the opposite vertices) respectively on the circumcircle of ABC. Let P be the midpoint of arc BC containing point A. Suppose lines BP and XZ meet at M , while lines CP and XY meet at N. Find the square of the distance from X to MN.

27. Let $p = 2^{16} + 1$ be a prime, and let S be the set of positive integers not divisible by p. Let $f : S \to \{0, 1, 2, \ldots, p - 1\}$ be a function satisfying

$$f(x)f(y) \equiv f(xy) + f(xy^{p-2}) \pmod{p} \quad \text{and} \quad f(x + p) = f(x)$$

for all $x, y \in S$. Let N be the product of all possible nonzero values of $f(81)$. Find the remainder when when N is divided by p.

28. Let S be the set of all pairs (a, b) of real numbers satisfying $1 + a + a^2 + a^3 = b^2(1 + 3a)$ and $1 + 2a + 3a^2 = b^2 - \frac{5}{b}$. Find $A + B + C$, where

$$A = \prod_{(a,b)\in S} a, \quad B = \prod_{(a,b)\in S} b, \quad \text{and} \quad C = \sum_{(a,b)\in S} ab.$$

29. Let ABC be a triangle with circumcenter O, incenter I, and circumcircle Γ. It is known that $AB = 7$, $BC = 8$, $CA = 9$. Let M denote the midpoint of major arc $\widehat{BAC}$ of Γ, and let D denote the intersection of Γ with the circumcircle of $\triangle IMO$ (other than M). Let E denote the reflection of D over line IO. Find the integer closest to $1000 \cdot \frac{BE}{CE}$.

30. Let $p = 2^{16} + 1$ be an odd prime. Define $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. Compute the remainder when

$$(p-1)! \sum_{n=1}^{p-1} H_n \cdot 4^n \cdot \binom{2p-2n}{p-n}$$

is divided by p.