The Online Math Open Spring Contest
March 23 – April 3, 2018

Online Math Open

Acknowledgements

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Contest Information

These rules supersede any rules found elsewhere about the OMO. Please send any further questions directly to the OMO Team at `OnlineMathOpenTeam@gmail.com`.

Team Registration and Eligibility

Students may compete in teams of up to four people, but no student can belong to more than one team. Participants must not have graduated from high school (or the equivalent secondary school institution in other countries). Teams need not remain the same between the Fall and Spring contests, and students are permitted to participate in whichever contests they like.
Only one member on each team needs to register an account on the website. Please check the website, `http://internetolympiad.org/pages/14-omo_info`, for registration instructions.
Note: when we say "up to four", we really do mean "up to"! Because the time limit is so long, partial teams are not significantly disadvantaged, and we welcome their participation.

Contest Format and Rules

The 2018 Spring Contest will consist of 30 problems; the answer to each problem will be an integer between 0 and $2^{31} - 1 = 2147483647$ inclusive. The contest window will be

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from 7PM ET on the start day to **7PM ET on the end day**. There is no time limit other than the contest window.

1. Four-function calculators (calculators which can perform only the four basic arithmetic operations) are permitted on the Online Math Open. **Any other computational aids, including scientific calculators, graphing calculators, or computer programs is prohibited.** All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.

2. Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids, such as Geogebra and graphing calculators, are not allowed. Print and electronic publications are also not allowed.

3. Members of different teams cannot communicate with each other about the contest while the contest is running.

4. Your score is the number of questions answered correctly; that is, every problem is worth one point. Ties will be broken based on the "hardest" problem that a team answered correctly. Remaining ties will be broken by the second hardest problem solved, and so on. (Problem m is harder than problem n if fewer teams solve problem m OR if the number of solves is equal and $m > n$.)

5. *Participation in the Online Math Open is free.*

Clarifications and Results

Clarifications will be posted as they are answered. For the most recent contests, they will be posted at `http://internetolympiad.org/pages/n/omo_problems`. If you have a question about problem wording, please email `OnlineMathOpenTeam@gmail.com` with "Clarification" in the subject. We have the right to deny clarification requests that we feel we cannot answer.
After the contest is over, we will release the answers to the problems within the next day. Please do not discuss the test until answers are released. If you have a protest about an answer, you may send an email to `OnlineMathOpenTeam@gmail.com`. (Include "Protest" in the subject). Results will be released in the following weeks. (Results will be counted only for teams that submit answers at least once. Teams that only register an account will not be listed in the final rankings.)

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1. Farmer James has three types of cows on his farm. A cow with zero legs is called a *ground beef*, a cow with one leg is called a *steak*, and a cow with two legs is called a *lean beef*. Farmer James counts a total of 20 cows and 18 legs on his farm. How many more *ground beef*s than *lean beef*s does Farmer James have?

2. The area of a circle (in square inches) is numerically larger than its circumference (in inches). What is the smallest possible integral area of the circle, in square inches?

3. Hen Hao randomly selects two distinct squares on a standard 8×8 chessboard. Given that the two squares touch (at either a vertex or a side), the probability that the two squares are the same color can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Find $100m + n$.

4. Define $f(x) = |x - 1|$. Determine the number of real numbers x such that $f(f(\cdots f(f(x))\cdots)) = 0$, where there are 2018 f's in the equation.

5. A mouse has a wheel of cheese which is cut into 2018 slices. The mouse also has a 2019-sided die, with faces labeled $0, 1, 2, \ldots, 2018$, and with each face equally likely to come up. Every second, the mouse rolls the dice. If the dice lands on k, and the mouse has at least k slices of cheese remaining, then the mouse eats k slices of cheese; otherwise, the mouse does nothing. What is the expected number of seconds until all the cheese is gone?

6. Let $f(x) = x^2 + x$ for all real x. There exist positive integers m and n, and distinct nonzero real numbers y and z, such that $f(y) = f(z) = m + \sqrt{n}$ and $f(\frac{1}{y}) + f(\frac{1}{z}) = \frac{1}{10}$. Compute $100m + n$.

7. A quadrilateral and a pentagon (both not self-intersecting) intersect each other at N distinct points, where N is a positive integer. What is the maximal possible value of N?

8. Compute the number of ordered quadruples (a, b, c, d) of distinct positive integers such that $\binom{\binom{a}{b}}{\binom{c}{d}} = 21$.

9. Let k be a positive integer. In the coordinate plane, circle ω has positive integer radius and is tangent to both axes. Suppose that ω passes through $(1, 1000 + k)$. Compute the smallest possible value of k.

10. The one hundred U.S. Senators are standing in a line in alphabetical order. Each senator either always tells the truth or always lies. The ith person in line says;

    "Of the $101 - i$ people who are not ahead of me in line (including myself), more than half of them are truth-tellers."

    How many possibilities are there for the set of truth-tellers on the U.S. Senate?

11. Lunasa, Merlin, and Lyrica are performing in a concert. Each of them will perform two different solos, and each pair of them will perform a duet, for nine distinct pieces in total. Since the performances are very demanding, no one is allowed to perform in two pieces in a row. In how many different ways can the pieces be arranged in this concert?

12. Near the end of a game of Fish, Celia is playing against a team consisting of Alice and Betsy. Each of the three players holds two cards in their hand, and together they have the Nine, Ten, Jack, Queen, King, and Ace of Spades (this set of cards is known by all three players). Besides the two cards she already has, each of them has no information regarding the other two's hands (In particular, teammates Alice and Betsy do not know each other's cards).

    It is currently Celia's turn. On a player's turn, the player must ask a player on the other team whether she has a certain card that is in the set of six cards but *not* in the asker's hand. If the player being asked does indeed have the card, then she must reveal the card and put it in the askeras hand, and the asker shall ask again (but may ask a different player on the other team); otherwise, she refuses and it is now her turn. Moreover, a card may not be asked if it is known (to the asker) to be not in the asked person's hand. The game ends when all six cards belong to one team, and the team with all the cards wins. Under optimal play, the probability that Celia wins the game is $\frac{p}{q}$ for relatively prime positive integers p and q. Find $100p + q$.

13. Find the smallest positive integer n for which the polynomial

$$x^n - x^{n-1} - x^{n-2} - \cdots - x - 1$$

has a real root greater than 1.999.

14. Let ABC be a triangle with $AB = 20$ and $AC = 18$. E is on segment AC and F is on segment AB such that $AE = AF = 8$. Let BE and CF intersect at G. Given that $AEGF$ is cyclic, then $BC = m\sqrt{n}$ for positive integers m and n such that n is not divisible by the square of any prime. Compute $100m+n$.

15. Let $\mathbb{N}$ denote the set of positive integers. For how many positive integers $k \leq 2018$ do there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = 2n$ for all $n \in \mathbb{N}$ and $f(k) = 2018$?

16. In a rectangular 57×57 grid of cells, k of the cells are colored black. What is the smallest positive integer k such that there must exist a rectangle, with sides parallel to the edges of the grid, that has its four vertices at the center of distinct black cells?

17. Let S be the set of all subsets of $\{2, 3, \ldots, 2016\}$ with size 1007, and for a nonempty set T of numbers, let $f(T)$ be the product of the elements in T. Determine the remainder when

$$\sum_{T \in S} \left(f(T) - f(T)^{-1}\right)^2$$

is divided by 2017. Note: For b relatively prime to 2017, we say that b^{-1} is the unique positive integer less than 2017 for which 2017 divides $bb^{-1} - 1$.

18. Suppose that a, b, c are real numbers such that $a < b < c$ and $a^3 - 3a + 1 = b^3 - 3b + 1 = c^3 - 3c + 1 = 0$. Then $\frac{1}{a^2+b} + \frac{1}{b^2+c} + \frac{1}{c^2+a}$ can be written as $\frac{p}{q}$ for relatively prime positive integers p and q. Find $100p + q$.

19. Let $P(x)$ be a polynomial of degree at most 2018 such that $P(i) = \binom{2018}{i}$ for all integer i such that $0 \leq i \leq 2018$. Find the largest nonnegative integer n such that $2^n \mid P(2020)$.

20. Let ABC be a triangle with $AB = 7, BC = 5$, and $CA = 6$. Let D be a variable point on segment BC, and let the perpendicular bisector of AD meet segments AC, AB at E, F, respectively. It is given that there is a point P inside $\triangle ABC$ such that $\frac{AP}{PC} = \frac{AE}{EC}$ and $\frac{AP}{PB} = \frac{AF}{FB}$. The length of the path traced by P as D varies along segment BC can be expressed as $\sqrt{\frac{m}{n}} \sin^{-1}\left(\sqrt{\frac{1}{7}}\right)$, where m and n are relatively prime positive integers, and angles are measured in radians. Compute $100m + n$.

21. Let $\bigoplus$ and $\bigotimes$ be two binary boolean operators, i.e. functions that send $\{\text{True}, \text{False}\} \times \{\text{True}, \text{False}\}$ to $\{\text{True}, \text{False}\}$. Find the number of such pairs $(\bigoplus, \bigotimes)$ such that $\bigoplus$ and $\bigotimes$ distribute over each other, that is, for any three boolean values a, b, c, the following four equations hold:

    (a) $c \bigotimes (a \bigoplus b) = (c \bigotimes a) \bigoplus (c \bigotimes b)$;
    (b) $(a \bigoplus b) \bigotimes c = (a \bigotimes c) \bigoplus (b \bigotimes c)$;
    (c) $c \bigoplus (a \bigotimes b) = (c \bigoplus a) \bigotimes (c \bigoplus b)$;
    (d) $(a \bigotimes b) \bigoplus c = (a \bigoplus c) \bigotimes (b \bigoplus c)$.

22. Let $p = 9001$ be a prime number and let $\mathbb{Z}/p\mathbb{Z}$ denote the additive group of integers modulo p. Furthermore, if $A, B \subset \mathbb{Z}/p\mathbb{Z}$, then denote $A + B = \{a + b \pmod{p} | a \in A, b \in B\}$. Let $s_1, s_2, \ldots, s_8$ are positive integers that are at least 2. Yang the Sheep notices that no matter how he chooses sets $T_1, T_2, \ldots, T_8 \subset \mathbb{Z}/p\mathbb{Z}$ such that $|T_i| = s_i$ for $1 \leq i \leq 8$, $T_1 + T_2 + \cdots + T_7$ is never equal to $\mathbb{Z}/p\mathbb{Z}$, but $T_1 + T_2 + \cdots + T_8$ must always be exactly $\mathbb{Z}/p\mathbb{Z}$. What is the minimum possible value of s_8?

23. Let ABC be a triangle with $BC = 13, CA = 11, AB = 10$. Let A_1 be the midpoint of BC. A variable line ℓ passes through A_1 and meets AC, AB at B_1, C_1. Let B_2, C_2 be points with $B_2B = B_2C, B_2C_1 \perp AB, C_2B = C_2C, C_2B_1 \perp AC$, and define $P = BB_2 \cap CC_2$. Suppose the circles of diameters BB_2, CC_2 meet at a point $Q \neq A_1$. Given that Q lies on the same side of line BC as A, the minimum possible value of $\dfrac{PB}{PC} + \dfrac{QB}{QC}$ can be expressed in the form $\dfrac{a\sqrt{b}}{c}$ for positive integers a, b, c with $\gcd(a, c) = 1$ and b squarefree. Determine $a + b + c$.

24. Find the number of ordered triples (a, b, c) of integers satisfying $0 \leq a, b, c \leq 1000$ for which

$$a^3 + b^3 + c^3 \equiv 3abc + 1 \pmod{1001}.$$

25. Let m and n be positive integers. Fuming Zeng gives James a rectangle, such that $m - 1$ lines are drawn parallel to one pair of sides and $n - 1$ lines are drawn parallel to the other pair of sides (with each line distinct and intersecting the interior of the rectangle), thus dividing the rectangle into an $m \times n$ grid of smaller rectangles. Fuming Zeng chooses $m + n - 1$ of the mn smaller rectangles and then tells James the area of each of the smaller rectangles. Of the $\dbinom{mn}{m+n-1}$ possible combinations of rectangles and their areas Fuming Zeng could have given, let $C_{m,n}$ be the number of combinations which would allow James to determine the area of the whole rectangle. Given that

$$A = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{m,n}\binom{m+n}{m}}{(m+n)^{m+n}},$$

then find the greatest integer less than $1000A$.

26. Let ABC be a triangle with incenter I. Let P and Q be points such that $IP \perp AC$, $IQ \perp AB$, and $IA \perp PQ$. Assume that BP and CQ intersect at the point $R \neq A$ on the circumcircle of ABC such that $AR \parallel BC$. Given that $\angle B - \angle C = 36^\circ$, the value of $\cos A$ can be expressed in the form $\frac{m-\sqrt{n}}{p}$ for positive integers m, n, p and where n is not divisible by the square of any prime. Find the value of $100m + 10n + p$.

27. Let $n = 2^{2018}$ and let $S = \{1, 2, ..., n\}$. For subsets $S_1, S_2, ..., S_n \subseteq S$, we call an ordered pair (i, j) *murine* if and only if $\{i, j\}$ is a subset of at least one of S_i, S_j. Then, a sequence of subsets $(S_1, \ldots, S_n)$ of S is called *tasty* if and only if:

    (a) For all i, $i \in S_i$.

    (b) For all i, $\displaystyle\bigcup_{j \in S_i} S_j = S_i$.

    (c) There do not exist pairwise distinct integers $a_1, a_2, ..., a_k$ with $k \geq 3$ such that for each i, (a_i, a_{i+1}) is murine, where indices are taken modulo k.

    (d) n divides $1 + |S_1| + |S_2| + ... + |S_n|$.

    Find the largest integer x such that 2^x divides the number of tasty sequences $(S_1, \ldots, S_n)$.

28. In $\triangle ABC$, the incircle ω has center I and is tangent to $\overline{CA}$ and $\overline{AB}$ at E and F respectively. The circumcircle of $\triangle BIC$ meets ω at P and Q. Lines AI and BC meet at D, and the circumcircle of $\triangle PDQ$ meets $\overline{BC}$ again at X. Suppose that $EF = PQ = 16$ and $PX + QX = 17$. Then BC^2 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

29. Let $q < 50$ be a prime number. Call a sequence of polynomials $P_0(x), P_1(x), P_2(x), ..., P_{q^2}(x)$ *tasty* if it satisfies the following conditions:

    - P_i has degree i for each i (where we consider constant polynomials, including the 0 polynomial, to have degree 0)

- The coefficients of P_i are integers between 0 and $q-1$ for each i.
- For any $0 \le i, j \le q^2$, the polynomial $P_i(P_j(x)) - P_j(P_i(x))$ has all its coefficients divisible by q.

As q varies over all such prime numbers, determine the total number of tasty sequences of polynomials.

30. Let $p = 2017$. Given a positive integer n, an $n \times n$ matrix A is formed with each element a_{ij} randomly selected, with equal probability, from $\{0, 1, ..., p-1\}$. Let q_n be probability that $\det A \equiv 1 \pmod{p}$. Let $q = \lim_{n\to\infty} q_n$. If $d_1, d_2, d_3, \ldots$ are the digits after the decimal point in the base p expansion of q, then compute the remainder when $\displaystyle\sum_{k=1}^{p^2} d_k$ is divided by 10^9.