The Online Math Open Spring Contest
March 24 - April 4, 2017

Acknowledgements

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Contest Information

These rules supersede any rules found elsewhere about the OMO. Please send any further questions directly to the OMO Team at `OnlineMathOpenTeam@gmail.com`.

Team Registration and Eligibility

Students may compete in teams of up to four people, but no student can belong to more than one team. Participants must not have graduated from high school (or the equivalent secondary school institution in other countries). Teams need not remain the same between the Fall and Spring contests, and students are permitted to participate in whichever contests they like.
Only one member on each team needs to register an account on the website. Please check the website, `http://internetolympiad.org/pages/14-omo_info`, for registration instructions.
Note: when we say "up to four", we really do mean "up to"! Because the time limit is so long, partial teams are not significantly disadvantaged, and we welcome their participation.

Contest Format and Rules

The 2017 Spring Contest will consist of 30 problems; the answer to each problem will be an integer between 0 and $2^{31} - 1 = 2147483647$ inclusive. The contest window will be

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from 7PM ET on the start day to **7PM ET on the end day**. There is no time limit other than the contest window.

1. Four-function calculators (calculators which can perform only the four basic arithmetic operations) are permitted on the Online Math Open. **Any other computational aids, including scientific calculators, graphing calculators, or computer programs is prohibited.** All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.

2. Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids, such as Geogebra and graphing calculators, are not allowed. Print and electronic publications are also not allowed.

3. Members of different teams cannot communicate with each other about the contest while the contest is running.

4. Your score is the number of questions answered correctly; that is, every problem is worth one point. Ties will be broken based on the "hardest" problem that a team answered correctly. Remaining ties will be broken by the second hardest problem solved, and so on. (Problem m is harder than problem n if fewer teams solve problem m OR if the number of solves is equal and $m > n$.)

5. *Participation in the Online Math Open is free.*

Clarifications and Results

Clarifications will be posted as they are answered. For the most recent contests, they will be posted at `http://internetolympiad.org/pages/n/omo_problems`. If you have a question about problem wording, please email `OnlineMathOpenTeam@gmail.com` with "Clarification" in the subject. We have the right to deny clarification requests that we feel we cannot answer.
After the contest is over, we will release the answers to the problems within the next day. Please do not discuss the test until answers are released. If you have a protest about an answer, you may send an email to `OnlineMathOpenTeam@gmail.com`. (Include "Protest" in the subject). Results will be released in the following weeks. (Results will be counted only for teams that submit answers at least once. Teams that only register an account will not be listed in the final rankings.)

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1. Find the smallest positive integer that is relatively prime to each of $2, 20, 204$, and 2048.

2. A positive integer n is called *bad* if it cannot be expressed as the product of two distinct positive integers greater than 1. Find the number of bad positive integers less than 100.

3. In rectangle $ABCD$, $AB = 6$ and $BC = 16$. Points P, Q are chosen on the interior of side AB such that $AP = PQ = QB$, and points R, S are chosen on the interior of side CD such that $CR = RS = SD$. Find the area of the region formed by the union of parallelograms $APCR$ and $QBSD$.

4. Lunasa, Merlin, and Lyrica each has an instrument. We know the following about the prices of their instruments:

   - If we raise the price of Lunasa's violin by 50% and decrease the price of Merlin's trumpet by 50%, the violin will be \$50 more expensive than the trumpet;
   - If we raise the price of Merlin's trumpet by 50% and decrease the price of Lyrica's piano by 50%, the trumpet will be \$50 more expensive than the piano.

   Given these conditions only, there exist integers m and n such that if we raise the price of Lunasa's violin by $m\%$ and decrease the price of Lyrica's piano by $m\%$, the violin must be exactly $\$n$ more expensive than the piano. Find $100m + n$.

5. There are 15 (not necessarily distinct) integers chosen uniformly at random from the range from 0 to 999, inclusive. Yang then computes the sum of their units digits, while Michael computes the last three digits of their sum. The probability of them getting the same result is $\frac{m}{n}$ for relatively prime positive integers m, n. Find $100m + n$.

6. Let $ABCDEF$ be a regular hexagon with side length 10 inscribed in a circle ω. X, Y, and Z are points on ω such that X is on minor arc AB, Y is on minor arc CD, and Z is on minor arc EF, where X may coincide with A or B (And similarly for Y and Z). Compute the square of the smallest possible area of XYZ.

7. Let S be the set of all positive integers between 1 and 2017, inclusive. Suppose that the least common multiple of all elements in S is L. Find the number of elements in S that do not divide $\frac{L}{2016}$.

8. A five-digit positive integer is called *k-phobic* if no matter how one chooses to alter at most four of the digits, the resulting number (after disregarding any leading zeroes) will not be a multiple of k. Find the smallest positive integer value of k such that there exists a k-phobic number.

9. Kevin is trying to solve an economics question which has six steps. At each step, he has a probability p of making a sign error. Let q be the probability that Kevin makes an even number of sign errors (thus answering the question correctly!). For how many values of $0 \le p \le 1$ is it true that $p + q = 1$?

10. When Cirno walks into her perfect math class today, she sees a polynomial $P(x) = 1$ (of degree 0) on the blackboard. As her teacher explains, for her pop quiz today, she will have to perform one of the two actions every minute:

    - Add a monomial to $P(x)$ so that the degree of P increases by 1 and P remains monic;
    - Replace the current polynomial $P(x)$ by $P(x+1)$. For example, if the current polynomial is $x^2 + 2x + 3$, then she will change it to $(x+1)^2 + 2(x+1) + 3 = x^2 + 4x + 6$.

    Her score for the pop quiz is the sum of coefficients of the polynomial at the end of 9 minutes. Given that Cirno (miraculously) doesn't make any mistakes in performing the actions, what is the maximum score that she can get?

11. Let a_1, a_2, a_3, a_4 be integers with distinct absolute values. In the coordinate plane, let $A_1 = (a_1, a_1^2)$, $A_2 = (a_2, a_2^2)$, $A_3 = (a_3, a_3^2)$ and $A_4 = (a_4, a_4^2)$. Assume that lines A_1A_2 and A_3A_4 intersect on the y-axis at an acute angle of θ. The maximum possible value for $\tan\theta$ can be expressed in the form $\frac{m}{n}$ for relative prime positive integers m and n. Find $100m + n$.

12. Alice has an isosceles triangle M_0N_0P, where $M_0P = N_0P$ and $\angle M_0PN_0 = \alpha^\circ$. (The angle is measured in degrees.) Given a triangle M_iN_jP for nonnegative integers i and j, Alice may perform one of two *elongations*:

    - an M-*elongation*, where she extends ray $\overrightarrow{PM_i}$ to a point M_{i+1} where $M_iM_{i+1} = M_iN_j$ and then removes the point M_i.
    - an N-*elongation*, where she extends ray $\overrightarrow{PN_j}$ to a point N_{j+1} where $N_jN_{j+1} = M_iN_j$ and then removes the point N_j.

    After a series of 5 elongations, k of which were M-elongations, Alice finds that triangle $M_kN_{5-k}P$ is an isosceles triangle. Given that 10α is an integer, compute 10α.

13. On a real number line, the points $1, 2, 3, \dots, 11$ are marked. A grasshopper starts at point 1, then jumps to each of the other 10 marked points in some order so that no point is visited twice, before returning to point 1. The maximal length that he could have jumped in total is L, and there are N possible ways to achieve this maximum. Compute $L + N$.

14. Let ABC be a triangle, not right-angled, with positive integer angle measures (in degrees) and circumcenter O. Say that a triangle ABC is *good* if the following three conditions hold:

    - There exists a point $P \neq A$ on side AB such that the circumcircle of $\triangle POA$ is tangent to BO.
    - There exists a point $Q \neq A$ on side AC such that the circumcircle of $\triangle QOA$ is tangent to CO.
    - The perimeter of $\triangle APQ$ is at least $AB + AC$.

    Determine the number of ordered triples $(\angle A, \angle B, \angle C)$ for which $\triangle ABC$ is good.

15. Let $\phi(n)$ denote the number of positive integers less than or equal to n which are relatively prime to n. Over all integers $1 \leq n \leq 100$, find the maximum value of $\phi(n^2 + 2n) - \phi(n^2)$.

16. Let S denote the set of subsets of $\{1, 2, \dots, 2017\}$. For two sets A and B of integers, define $A \circ B$ as the *symmetric difference* of A and B. (In other words, $A \circ B$ is the set of integers that are an element of exactly one of A and B.) Let N be the number of functions $f : S \to S$ such that $f(A \circ B) = f(A) \circ f(B)$ for all $A, B \in S$. Find the remainder when N is divided by 1000.

17. Let ABC be a triangle with $BC = 7, AB = 5$, and $AC = 8$. Let M, N be the midpoints of sides AC, AB respectively, and let O be the circumcenter of ABC. Let BO, CO meet AC, AB at P and Q, respectively. If MN meets PQ at R and OR meets BC at S, then the value of OS^2 can be written in the form $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $100m + n$.

18. Let p be an odd prime number less than 10^5. Granite and Pomegranate play a game. First, Granite picks a integer $c \in \{2, 3, \dots, p-1\}$. Pomegranate then picks two integers d and x, defines $f(t) = ct + d$, and writes x on a sheet of paper. Next, Granite writes $f(x)$ on the paper, Pomegranate writes $f(f(x))$, Granite writes $f(f(f(x)))$, and so on, with the players taking turns writing. The game ends when two numbers appear on the paper whose difference is a multiple of p, and the player who wrote the most recent number wins. Find the sum of all p for which Pomegranate has a winning strategy.

19. For each integer $1 \leq j \leq 2017$, let S_j denote the set of integers $0 \leq i \leq 2^{2017} - 1$ such that $\left\lfloor \frac{i}{2^{j-1}} \right\rfloor$ is an odd integer. Let P be a polynomial such that

$$P\left(x_0, x_1, \ldots, x_{2^{2017}-1}\right) = \prod_{1 \le j \le 2017} \left(1 - \prod_{i \in S_j} x_i\right).$$

Compute the remainder when

$$\sum_{\left(x_0, \ldots, x_{2^{2017}-1}\right) \in \{0,1\}^{2^{2017}}} P\left(x_0, \ldots, x_{2^{2017}-1}\right)$$

is divided by 2017.

20. Let n be a fixed positive integer. For integer m satisfying $|m| \le n$, define $S_m = \sum_{\substack{i-j=m \\ 0 \le i,j \le n}} \frac{1}{2^{i+j}}$. Then

$$\lim_{n \to \infty} \left(S_{-n}^2 + S_{-n+1}^2 + \ldots + S_n^2\right)$$

can be expressed in the form $\frac{p}{q}$ for relatively prime positive integers p, q. Compute $100p + q$.

21. Let $\mathbb{Z}_{\ge 0}$ be the set of nonnegative integers. Let $f : \mathbb{Z}_{\ge 0} \to \mathbb{Z}_{\ge 0}$ be a function such that, for all $a, b \in \mathbb{Z}_{\ge 0}$:

$$f(a)^2 + f(b)^2 + f(a+b)^2 = 1 + 2f(a)f(b)f(a+b).$$

Furthermore, suppose there exists $n \in \mathbb{Z}_{\ge 0}$ such that $f(n) = 577$. Let S be the sum of all possible values of $f(2017)$. Find the remainder when S is divided by 2017.

22. Let $S = \{(x, y) \mid -1 \le xy \le 1\}$ be a subset of the real coordinate plane. If the smallest real number that is greater than or equal to the area of any triangle whose interior lies entirely in S is A, compute the greatest integer not exceeding $1000A$.

23. Determine the number of ordered quintuples (a, b, c, d, e) of integers with $0 \le a < b < c < d < e \le 30$ for which there exist polynomials $Q(x)$ and $R(x)$ with integer coefficients such that

$$x^a + x^b + x^c + x^d + x^e = Q(x)(x^5 + x^4 + x^2 + x + 1) + 2R(x).$$

24. For any positive integer n, let S_n denote the set of positive integers which cannot be written in the form $an + 2017b$ for nonnegative integers a and b. Let A_n denote the average of the elements of S_n if the cardinality of S_n is positive and finite, and 0 otherwise. Compute

$$\left\lfloor \sum_{n=1}^{\infty} \frac{A_n}{2^n} \right\rfloor.$$

25. A *simple hyperplane* in $\mathbb{R}^4$ has the form

$$k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = 0$$

for some integers $k_1, k_2, k_3, k_4 \in \{-1, 0, 1\}$ that are not all zero. Find the number of regions that the set of all simple hyperplanes divide the unit ball $x_1^2 + x_2^2 + x_3^2 + x_4^2 \le 1$ into.

26. Let ABC be a triangle with $AB = 13, BC = 15, AC = 14$, circumcenter O, and orthocenter H, and let M, N be the midpoints of minor and major arcs BC on the circumcircle of ABC. Suppose $P \in AB, Q \in AC$ satisfy that P, O, Q are collinear and $PQ || AN$, and point I satisfies $IP \perp AB, IQ \perp AC$. Let H' be the reflection of H over line PQ, and suppose $H'I$ meets PQ at a point T. If $\frac{MT}{NT}$ can be written in the form $\frac{\sqrt{m}}{n}$ for positive integers m, n where m is not divisible by the square of any prime, then find $100m + n$.

27. Let N be the number of functions $f : \mathbb{Z}/16\mathbb{Z} \to \mathbb{Z}/16\mathbb{Z}$ such that for all $a, b \in \mathbb{Z}/16\mathbb{Z}$:

$$f(a)^2 + f(b)^2 + f(a+b)^2 \equiv 1 + 2f(a)f(b)f(a+b) \pmod{16}.$$

Find the remainder when N is divided by 2017.

28. Let S denote the set of fractions $\frac{m}{n}$ for relatively prime positive integers m and n with $m+n \le 10000$. The least fraction in S that is strictly greater than

$$\prod_{i=0}^{\infty}\left(1 - \frac{1}{10^{2i+1}}\right)$$

can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $1000p + q$.

29. Let ABC be a triangle with $AB = 2\sqrt{6}, BC = 5, CA = \sqrt{26}$, midpoint M of BC, circumcircle Ω, and orthocenter H. Let BH intersect AC at E and CH intersect AB at F. Let R be the midpoint of EF and let N be the midpoint of AH. Let AR intersect the circumcircle of AHM again at L. Let the circumcircle of ANL intersect Ω and the circumcircle of BNC at J and O, respectively. Let circles AHM and JMO intersect again at U, and let AU intersect the circumcircle of AHC again at $V \neq A$. The square of the length of CV can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Find $100m + n$.

30. Let $p = 2017$ be a prime. Given a positive integer n, let T be the set of all $n \times n$ matrices with entries in $\mathbb{Z}/p\mathbb{Z}$. A function $f : T \to \mathbb{Z}/p\mathbb{Z}$ is called an *n-determinant* if for every pair $1 \le i, j \le n$ with $i \neq j$,

$$f(A) = f(A'),$$

where A' is the matrix obtained by adding the jth row to the ith row.

Let a_n be the number of n-determinants. Over all $n \ge 1$, how many distinct remainders of a_n are possible when divided by $\frac{(p^p - 1)(p^{p-1} - 1)}{p - 1}$?