NIMO
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Contest **14**
May **2014**

May 15, 2014
8:00 PM – 8:40 PM ET

Problem 1. Let A, B, C, D be four points on a line in this order. Suppose that $AC = 25$, $BD = 40$, and $AD = 57$. Compute $AB \cdot CD + AD \cdot BC$.

Problem 2. In the Generic Math Tournament, 99 people participate. One of the participants, Alfred, scores 16th in Algebra, 30th in Combinatorics, and 23rd in Geometry (and does not tie with anyone). The overall ranking is computed by adding the scores from all three tests. Given this information, let B be the best ranking that Alfred could have achieved, and let W be the worst ranking that he could have achieved. Compute $100B + W$.

Problem 3. In triangle ABC, we have $AB = AC = 20$ and $BC = 14$. Consider points M on $\overline{AB}$ and N on $\overline{AC}$. If the minimum value of the sum $BN + MN + MC$ is x, compute $100x$.

Problem 4. Define the infinite products

$$A = \prod_{i=2}^{\infty} \left(1 - \frac{1}{n^3}\right) \text{ and } B = \prod_{i=1}^{\infty} \left(1 + \frac{1}{n(n+1)}\right).$$

If $\frac{A}{B} = \frac{m}{n}$ where m, n are relatively prime positive integers, determine $100m + n$.

Problem 5. Find the largest integer n for which 2^n divides

$$\binom{2}{1}\binom{4}{2}\binom{6}{3}\cdots\binom{128}{64}.$$

Problem 6. 10 students are arranged in a row. Every minute, a new student is inserted in the row (which can occur in the front and in the back as well, hence 11 possible places) with a uniform $\frac{1}{11}$ probability of each location. Then, either the frontmost or the backmost student is removed from the row (each with a $\frac{1}{2}$ probability).

Suppose you are the eighth in the line from the front. The probability that you exit the row from the front rather than the back is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Problem 7. Ana and Banana play a game. First, Ana picks a real number p with $0 \le p \le 1$. Then, Banana picks an integer h greater than 1 and creates a spaceship with h hit points. Now every minute, Ana decreases the spaceship's hit points by 2 with probability $1 - p$, and by 3 with probability p. Ana wins if and only if the number of hit points is reduced to exactly 0 at some point (in particular, if the spaceship has a negative number of hit points at any time then Ana loses). Given that Ana and Banana select p and h optimally, compute the integer closest to $1000p$.

Problem 8. Let x be a positive real number. Define

$$A = \sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)!}, \quad B = \sum_{k=0}^{\infty} \frac{x^{3k+1}}{(3k+1)!}, \quad \text{and} \quad C = \sum_{k=0}^{\infty} \frac{x^{3k+2}}{(3k+2)!}.$$

Given that $A^3 + B^3 + C^3 + 8ABC = 2014$, compute ABC.

Time limit: 40 minutes
Maximum score is 56 points