NIMO Monthly Contest

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Revenge 13   8:00 PM – 8:40 PM ET

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1. Let $\eta(m)$ be the product of all positive integers that divide m, including 1 and m. If $\eta(\eta(\eta(10))) = 10^n$, compute n.

2. Two points A and B are selected independently and uniformly at random along the perimeter of a unit square with vertices at $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. The probability that the y-coordinate of A is strictly greater than the y-coordinate of B can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $100m + n$.

3. Find the number of positive integers n with exactly 1974 factors such that no prime greater than 40 divides n, and n ends in one of the digits 1, 3, 7, 9. (Note that $1974 = 2 \cdot 3 \cdot 7 \cdot 47$.)

4. A black bishop and a white king are placed randomly on a 2000×2000 chessboard (in distinct squares). Let p be the probability that the bishop attacks the king (that is, the bishop and king lie on some common diagonal of the board). Then p can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute m.

5. Let a positive integer n be *nice* if there exists a positive integer m such that

$$n^3 < 5mn < n^3 + 100.$$

   Find the number of *nice* positive integers.

6. Let $P(x)$ be a polynomial with real coefficients such that $P(12) = 20$ and

$$(x-1) \cdot P(16x) = (8x-1) \cdot P(8x)$$

   holds for all real numbers x. Compute the remainder when $P(2014)$ is divided by 1000.

7. Let N denote the number of ordered pairs of sets (A, B) such that $A \cup B$ is a size-999 subset of $\{1, 2, \ldots, 1997\}$ and $(A \cap B) \cap \{1, 2\} = \{1\}$. If m and k are integers such that $3^m 5^k$ divides N, compute the the largest possible value of $m + k$.

8. Triangle ABC lies entirely in the first quadrant of the Cartesian plane, and its sides have slopes 63, 73, 97. Suppose the curve $\mathcal{V}$ with equation $y = (x+3)(x^2+3)$ passes through the vertices of ABC. Find the sum of the slopes of the three tangents to $\mathcal{V}$ at each of A, B, C.