NIMO Monthly Contest

www.internetolympiad.org

Contest 11   8:00 PM – 8:40 PM ET

January 31, 2014

1. Define $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Let the sum of all H_n that are terminating in base 10 be S. If $S = m/n$ where m and n are relatively prime positive integers, find $100m + n$.

2. In the game of Guess the Card, two players each have a $\frac{1}{2}$ chance of winning and there is exactly one winner. Sixteen competitors stand in a circle, numbered $1, 2, \ldots, 16$ clockwise. They participate in an 4-round single-elimination tournament of Guess the Card. Each round, the referee randomly chooses one of the remaining players, and the players pair off going clockwise, starting from the chosen one; each pair then plays Guess the Card and the losers leave the circle. If the probability that players 1 and 9 face each other in the last round is $\frac{m}{n}$ where m, n are positive integers, find $100m + n$.

3. Call an integer k *debatable* if the number of odd factors of k is a power of two. What is the largest positive integer n such that there exists n consecutive debatable numbers? (Here, a power of two is defined to be any number of the form 2^m, where m is a nonnegative integer.)

4. Let a, b, c be positive reals for which

$$(a+b)(a+c) = bc + 2$$
$$(b+c)(b+a) = ca + 5$$
$$(c+a)(c+b) = ab + 9$$

   If $abc = \frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

5. In triangle ABC, $\sin A \sin B \sin C = \frac{1}{1000}$ and $AB \cdot BC \cdot CA = 1000$. What is the area of triangle ABC?

6. Suppose we wish to pick a random integer between 1 and N inclusive by flipping a fair coin. One way we can do this is through generating a random binary decimal between 0 and 1, then multiplying the result by N and taking the ceiling. However, this would take an infinite amount of time. We therefore stopping the flipping process after we have enough flips to determine the ceiling of the number. For instance, if $N = 3$, we could conclude that the number is 2 after flipping $.011_2$, but $.010_2$ is inconclusive.

   Suppose $N = 2014$. The expected number of flips for such a process is $\frac{m}{n}$ where m, n are relatively prime positive integers, find $100m + n$.

7. Let $P(n)$ be a polynomial of degree m with integer coefficients, where $m \leq 10$. Suppose that $P(0) = 0$, $P(n)$ has m distinct integer roots, and $P(n) + 1$ can be factored as the product of two nonconstant polynomials with integer coefficients. Find the sum of all possible values of $P(2)$.

8. The side lengths of $\triangle ABC$ are integers with no common factor greater than 1. Given that $\angle B = 2\angle C$ and $AB < 600$, compute the sum of all possible values of AB.