NIMO Monthly Contest

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Contest 10 8:00 PM – 8:40 PM ET

December 3, 2013

1. Richard likes to solve problems from the IMO Shortlist. In 2013, Richard solves 5 problems each Saturday and 7 problems each Sunday. He has school on weekdays, so he "only" solves 2, 1, 2, 1, 2 problems on each Monday, Tuesday, Wednesday, Thursday, and Friday, respectively – with the exception of December 3, 2013, where he solved 60 problems out of boredom. Altogether, how many problems does Richard solve in 2013?

2. How many integers n are there such that $(n+1!)(n+2!)(n+3!)\cdots(n+2013!)$ is divisible by 210 and $1 \le n \le 210$?

3. At Stanford in 1988, human calculator Shakuntala Devi was asked to compute $m = \sqrt[3]{61{,}629{,}875}$ and $n = \sqrt[7]{170{,}859{,}375}$. Given that m and n are both integers, compute $100m + n$.

4. Let $S = \{1, 2, \cdots, 2013\}$. Let N denote the number 9-tuples of sets $(S_1, S_2, \dots, S_9)$ such that $S_{2n-1}, S_{2n+1} \subseteq S_{2n} \subseteq S$ for $n = 1, 2, 3, 4$. Find the remainder when N is divided by 1000.

5. In a certain game, Auntie Hall has four boxes B_1, B_2, B_3, B_4, exactly one of which contains a valuable gemstone; the other three contain cups of yogurt. You are told the probability the gemstone lies in box B_n is $\frac{n}{10}$ for $n = 1, 2, 3, 4$.

   Initially you may select any of the four boxes; Auntie Hall then opens one of the other three boxes at random (which may contain the gemstone) and reveals its contents. Afterwards, you may change your selection to any of the four boxes, and you win if and only if your final selection contains the gemstone. Let the probability of winning assuming optimal play be $\frac{m}{n}$, where m and n are relatively prime integers. Compute $100m + n$.

6. Given a regular dodecagon (a convex polygon with 12 congruent sides and angles) with area 1, there are two possible ways to dissect this polygon into 12 equilateral triangles and 6 squares. Let T_1 denote the union of all triangles in the first dissection, and S_1 the union of all squares. Define T_2 and S_2 similarly for the second dissection. Let S and T denote the areas of $S_1 \cap S_2$ and $T_1 \cap T_2$, respectively. If $\frac{S}{T} = \frac{a+b\sqrt{3}}{c}$ where a and b are integers, c is a positive integer, and $\gcd(a, c) = 1$, compute $10000a + 100b + c$.

7. Let $ABCD$ be a convex quadrilateral for which $DA = AB$ and $CA = CB$. Set $I_0 = C$ and $J_0 = D$, and for each nonnegative integer n, let I_{n+1} and J_{n+1} denote the incenters of $\triangle I_n AB$ and $\triangle J_n AB$, respectively. Suppose that

   $$\angle DAC = 15^\circ, \quad \angle BAC = 65^\circ \quad \text{and} \quad \angle J_{2013}J_{2014}I_{2014} = \left(90 + \frac{2k+1}{2^n}\right)^\circ$$

   for some nonnegative integers n and k. Compute $n + k$.

8. The number $\frac{1}{2}$ is written on a blackboard. For a real number c with $0 < c < 1$, a *c-splay* is an operation in which every number x on the board is erased and replaced by the two numbers cx and $1 - c(1 - x)$. A *splay-sequence* $C = (c_1, c_2, c_3, c_4)$ is an application of a c_i-splay for $i = 1, 2, 3, 4$ in that order, and its *power* is defined by $P(C) = c_1 c_2 c_3 c_4$.

   Let S be the set of splay-sequences which yield the numbers $\frac{1}{17}, \frac{2}{17}, \dots, \frac{16}{17}$ on the blackboard in some order. If $\sum_{C \in S} P(C) = \frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.