NIMO Monthly Contest

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Revenge 9 8:00 PM – 8:40 PM ET

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1. A sequence $a_0, a_1, a_2, \dots$ of real numbers satisfies $a_0 = 999$, $a_1 = -999$, and $a_n = a_{n-1}a_{n+1}$ for each positive integer n. Compute $|a_1 + a_2 + \cdots + a_{1000}|$.

2. Let f be a non-constant polynomial such that
$$f(x-1) + f(x) + f(x+1) = \frac{f(x)^2}{2013x}$$
for all nonzero real numbers x. Find the sum of all possible values of $f(1)$.

3. Let $a_1, a_2, \dots, a_{1000}$ be positive integers whose sum is S. If $a_n!$ divides n for each $n = 1, 2, \dots, 1000$, compute the maximum possible value of S.

4. Consider a set of 1001 points in the plane, no three collinear. Compute the minimum number of segments that must be drawn so that among any four points, we can find a triangle.

5. Let d and n be positive integers such that d divides n, $n > 1000$, and n is not a perfect square. The minimum possible value of $|d - \sqrt{n}|$ can be written in the form $a\sqrt{b}+c$, where b is a positive integer not divisible by the square of any prime, and a and c are nonzero integers (not necessarily positive). Compute $a + b + c$.

6. Let ABC be a triangle with $AB = 42$, $AC = 39$, $BC = 45$. Let E, F be on the sides $\overline{AC}$ and $\overline{AB}$ such that $AF = 21, AE = 13$. Let $\overline{CF}$ and $\overline{BE}$ intersect at P, and let ray AP meet $\overline{BC}$ at D. Let O denote the circumcenter of $\triangle DEF$, and R its circumradius. Compute $CO^2 - R^2$.

7. Tyler has two calculators, both of which initially display zero. The first calculators has only two buttons, [+1] and [×2]. The second has only the buttons [+1] and [×4]. Both calculators update their displays immediately after each keystroke.

   A positive integer n is called *ambivalent* if the minimum number of keystrokes needed to display n on the first calculator equals the minimum number of keystrokes needed to display n on the second calculator. Find the sum of all ambivalent integers between 256 and 1024 inclusive.

8. Let $ABCD$ be a convex quadrilateral with $\angle ABC = 120°$ and $\angle BCD = 90°$, and let M and N denote the midpoints of $\overline{BC}$ and $\overline{CD}$. Suppose there exists a point P on the circumcircle of $\triangle CMN$ such that ray MP bisects $\overline{AD}$ and ray NP bisects $\overline{AB}$. If $AB + BC = 444$, $CD = 256$ and $BC = \frac{m}{n}$ for some relatively prime positive integers m and n, compute $100m + n$.