5th Annual April Fun Round

http://www.internetolympiad.org

April 1, 2017

Welcome to the 2017 April Fun Round! You may use any aids such as calculators, computers, Wikipedia, previous NIMO fun rounds, etc. You are also free to collaborate with other students (spread the fun!) but we ask that you do so privately to give everyone a chance at the problems (say, email or chats as opposed to public forums).

You have the whole day to work on the problems. Moreover, note that you are allowed five entries per problem instead of the usual three.

Important: each answer is a *nonnegative* integer not exceeding $2^{32}-1 = 4294967295$.

1 Maximal Cutoff (Evan Chen)

A finite set X of students, $|X| \geq 1$, participate in a qualifying exam. This exam consists of 25 questions, and on each question a student can earn 0, 1.5 or 6 points. A student S passes the exam if *either*

- ~~S scores at least 100 points, or~~
- S scores at least as high as 95% of the students in X.

Finally, the **cutoff** of the exam is defined as

$$C = \min_{S \in X \text{ passing}} (\text{score of } S).$$

What is the largest possible value of C?

2 Hackerz (Michael Tang)

Leah is solving problem #3 on the 2016 NEMO (National Electronic Math Olympiad), in which she has to solve an equation for the variable x. The problem asks her to input the quantity $100m + n$, where her answer is $x = m + \sqrt{n}$ for positive integers m and n. Leah finds the correct value of x and inputs the right answer.

A few minutes later, Kim, who is also taking the NEMO, hacks into Leah's account to copy her answers! She finds that Leah got the (disgusting) answer 2,319,123 for problem #3. After copying this answer, though, Kim notes that simply knowing Leah's final answer is not enough to determine x: that is, there are many possible values of x that all lead to the same final answer.

Of these possible values, suppose the largest one is $x = a + \sqrt{b}$ and the smallest one is $x = c + \sqrt{d}$, for positive integers a, b, c, d. What is $100(a + c) + (b + d)$?

3 Geometry (Michael Tang)

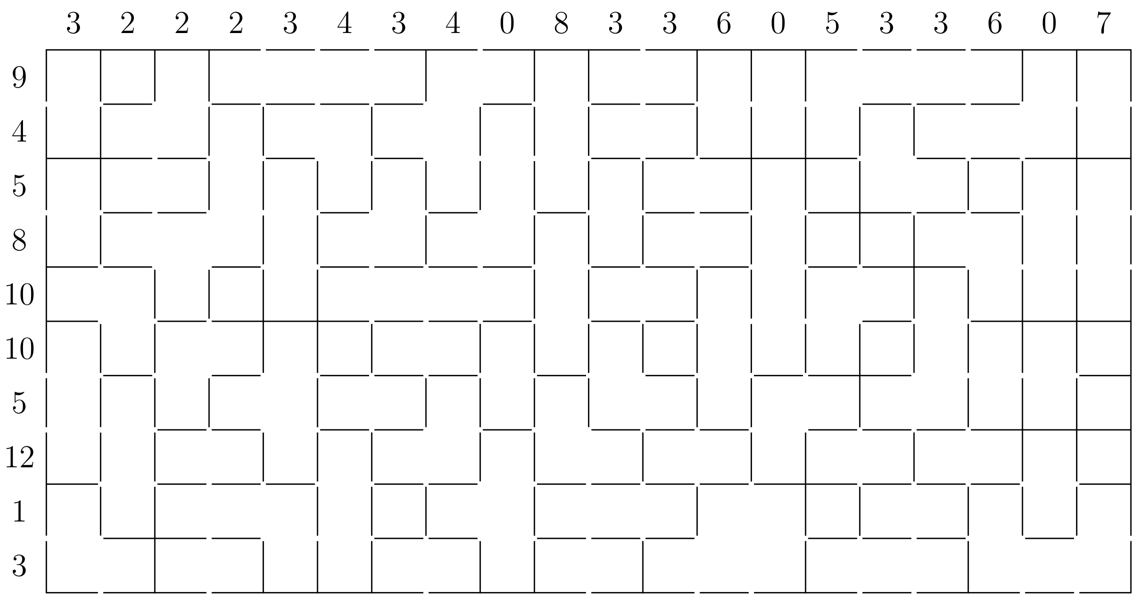
Let ω be the circumcircle of an equilateral triangle ABC with area 400π. To the nearest integer, what is the length of AB?

4 There's a First Time for Everything (Lewis Chen)

Let P be the chronologically earliest published math problem whose answer is 2,017,004,001. You are given that P exists. If P was published in the a-th year, b-th month, and c-th day (when measured using the Gregorian calendar and the GMT time zone), compute $1000000a + 1000b + c$.

5 Puzzle (Michael Tang)

The grid below is divided into regions along the grid lines. Completely shade some of the regions, leaving the other regions completely unshaded, such that each number to the left of the grid gives the number of shaded unit squares in its row, and each number above the grid gives the number of shaded unit squares in its column.



6 April $f$un Round (Michael Tang)

Find $f(4{,}000)^2 + f(40{,}000)^2 + f(400{,}000)^2 + f(4{,}000{,}000)^2$, where $f : \mathbb{Z} \to \mathbb{Z}$ is a function satisfying the following conditions:

1. For every positive integer x, we have $f(-x) = f(x) + 8$.
2. For $x \in \{3, 4, \ldots, 9\}$, we have
$$f(x+10) - f(10x) = 2 - \tfrac{1}{120}(x-3)(x-5)(x-6)(x-7)(x-8)(x-9).$$
3. $f(10^{12}) \in \{10, 11\}$.

4. For all $x \in \mathbb{Z}$, the sequence

$$\{f^n(x)\}_{n \geq 0} = x, f(x), f(f(x)), f(f(f(x))), \ldots$$

eventually becomes constant at 4.

7 Number Grid (David Altizio)

I don't know if MOP experience will help you ace this quiz....

	1.				2.					3.				
											4.			5.
			6.					7.			8.			
9.		10.											11.	
			12.				13.					14.		

Across

1. Compute the least composite positive integer N such that the sum of all prime numbers that divide N is 37.

2. David and Michael are playing a game. David has in mind a polynomial P with real coefficients, positive leading coefficient, and degree 100; Michael is aware of this information. Michael and David then alternate giving information via the following rule: Michael gives a real number t, and David returns the value of $P(t)^2$. What is the smallest number of calls necessary so that Michael can determine the polynomial P?

4. What is the largest number that is divisible by all numbers less than its square root?

6. Compute the least possible value of $m + 100n$, where m and n are positive integers such that
$$\frac{1^2 + 2^2 + \cdots + m^2}{1^2 + 2^2 + \cdots + n^2} = \frac{31}{254}.$$

7. For rational numbers a and b with $a > b$, define the *fractional average* of a and b to be the unique rational number c with the following property: when c is written in lowest terms, there exists an integer N such that adding N to both the numerator and denominator of c gives a, and subtracting N from both the numerator and denominator of c gives b. Suppose the fractional average of $\frac{1}{7}$ and $\frac{1}{10}$ is $\frac{m}{n}$, where m, n are coprime positive integers. What is $100m + n$?

12. Nonzero real numbers a, b, c satisfy the equations $a^2 + b^2 + c^2 = 2915$ and $(a-1)(b-1)(c-1) = abc - 1$. Compute $a + b + c$.

13. Two neighboring towns, MWMTown and NIMOTown, have a strange relationship with regard to weather. On a certain day, the probability that it is sunny in either town is $\frac{1}{23}$ greater than the probability of MWMTown being sunny, and the probability that it is sunny in MWMTown, given that it is sunny in NIMOTown, is $\frac{12}{23}$. If the probability that it is sunny in NIMOTown is $\frac{p}{q}$, for coprime positive integers p, q, what is $100p + q$?

14. Let $ABCDE$ be a convex pentagon inscribed inside a circle of radius 38. Let I_1 and I_2 denote the incenters of $\triangle ABD$ and $\triangle EBD$ respectively. Suppose that $AE = 4\sqrt{37}$ and $BC = CD = 57$. Over all such pentagons, compute the square of the maximum possible value of I_1I_2.

Down

3. How many positive integers n are there such that $n+3$ and n^2+3 are both perfect cubes?

5. There is a positive integer N, with two base-ten digits, such that N has its digits reversed when expressed in base 16. What is N?

8. Triangle $\triangle ABC$ has $BC = 6$, $AB = 9$, and $AC = 10$. Let H be its orthocenter. Suppose the circle with diameter AH intersects AB and AC for the second time at X and Y respectively. Then XY can be written in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. What is $m - n$?

9. Let $\mathcal{S}$ denote the set of all positive integers which do not contain the letter 'e' when written in English. What is the sum of the digits of the fourth smallest element of $\mathcal{S}$?

10. Let N denote the number of positive integers n between 1 and 1000 inclusive such that $\binom{3n}{n}$ is odd. What are the last two digits of N?

11. Twenty lamps are placed on the vertices of a regular 20-gon. The lights turn on and off according to the following rule: if, in second k the two lights adjacent to a light L are either both on or both off, L is on in second $k+1$; otherwise, L is off in second $k+1$. Let N denote the number of possible original settings of the lamps so that after four moves all lamps are on. What is the sum of the digits of N?

8 Personification (David Altizio)

The mathematician Daniel Tammet once said, "It sounds silly, but numbers are my friends." This is a strong statement, but I think it can be taken further. Numbers are humans. They live and interact with each other just like we do. Of course, they don't always get along - they have lots of different personalities, after all. But, like us, they also try to find that special one - that one for which they share a lot in common.