NIMO

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National Internet Math Olympiad

Contest 30

February 16, 2017
8:00 PM – 8:40 PM ET

Calculators and other computational aids are NOT allowed.

Problem 1. Suppose a, b and c are positive integers with the property that ab, bc, and ac are pairwise distinct perfect squares. What is the smallest possible value of $a+b+c$?

Problem 2. David draws a 2×2 grid of squares in chalk on the sidewalk outside NIMO HQ. He then draws one arrow in each square, each pointing in one of the four cardinal directions (north, south, east, west) parallel to the sides of the grid. In how many ways can David draw his arrows such that no two of the arrows are pointing at each other?

Problem 3. Compute the number of ordered quadruples of complex numbers (a, b, c, d) such that

$$(ax+by)^3 + (cx+dy)^3 = x^3 + y^3$$

holds for all complex numbers x, y.

Problem 4. A *divisibility chain* is a sequence of positive integers $(a_1, a_2, \ldots, a_n)$ such that a_k divides a_{k+1} for all $1 \le k < n$. Compute the number of divisibility chains of the form $(a, b, a^2, c, a^3, 360^9)$.

Problem 5. In triangle ABC, $AB = 12$, $BC = 17$, and $AC = 25$. Distinct points M and N lie on the circumcircle of ABC such that $BM = CM$ and $BN = CN$. If $AM + AN = \frac{a\sqrt{b}}{c}$, where a, b, c are positive integers such that $\gcd(a, c) = 1$ and b is not divisible by the square of a prime, compute $100a + 10b + c$.

Problem 6. Let n be a positive integer, and let $S_n = \{1, 2, \ldots, n\}$. For a permutation σ of S_n and an integer $a \in S_n$, let $d(a)$ be the least positive integer d for which

$$\underbrace{\sigma(\sigma(\ldots\sigma(a)\ldots))}_{d \text{ applications of } \sigma} = a$$

(or -1 if no such integer exists). Compute the value of n for which there exists a permutation σ of S_n satisfying the equations

$$d(1) + d(2) + \ldots + d(n) = 2017,$$
$$\frac{1}{d(1)} + \frac{1}{d(2)} + \ldots + \frac{1}{d(n)} = 2.$$

Problem 7. Let ABC be a triangle with $AB = 4$, $AC = 5$, $BC = 6$, and circumcircle Ω. Points E and F lie on AC and AB respectively such that $\angle ABE = \angle CBE$ and $\angle ACF = \angle BCF$. The second intersection point of the circumcircle of $\triangle AEF$ with Ω (other than A) is P. Suppose $AP^2 = \frac{m}{n}$ where m and n are positive relatively prime integers. Find $100m + n$.

Problem 8. The Fibonacci numbers $F_1, F_2, F_3, \ldots$ are defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for each integer $n \ge 1$. Let P be the unique polynomial of least degree for which $P(n) = F_n$ for all integers $1 \le n \le 10$. Compute the integer m for which

$$P(100) - \sum_{k=11}^{98} P(k) = \frac{m}{10}\binom{98}{9} + 144.$$

Time limit: 40 minutes
Maximum score is 56 points