November 30, 2016
8:00 PM – 8:40 PM ET

Calculators and other computational aids are NOT allowed.

Problem 1. In how many ways can Eve fill each of the six squares of a 2×3 grid with either a 0 or a 1, such that Anne can then divide the grid into three congruent rectangles: one containing two 0s, one containing two 1s, and one containing a 0 and a 1?

Problem 2. Let $\{a_n\}$ be a sequence of integers such that $a_1 = 2016$ and

$$\frac{a_{n-1} + a_n}{2} = n^2 - n + 1$$

for all $n \geq 1$. Compute a_{100}.

Problem 3. A circle C_0 is inscribed in an equilateral triangle XYZ of side length 112. Then, for each positive integer n, circle C_n is inscribed in the region bounded by XY, XZ, and an arc of circle C_{n-1}, forming an infinite sequence of circles tangent to sides XY and XZ and approaching vertex X. If these circles collectively have area $m\pi$, find m.

Problem 4. For how many positive integers $100 < n \leq 10000$ does $\lfloor \sqrt{n - 100} \rfloor$ divide n?

Problem 5. Let $\{a_i\}_{i=0}^{\infty}$ be a sequence of real numbers such that

$$\sum_{n=1}^{\infty} \frac{x^n}{1 - x^n} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

for all $|x| < 1$. Find a_{1000}.

Problem 6. Triangle $\triangle ABC$ has circumcenter O and incircle γ. Suppose that $\angle BAC = 60^\circ$ and O lies on γ. If

$$\tan B \tan C = a + \sqrt{b}$$

for positive integers a and b, compute $100a + b$.

Problem 7. Eve randomly chooses two **distinct** points on the coordinate plane from the set of all 11^2 lattice points (x, y) with $0 \leq x \leq 10$, $0 \leq y \leq 10$. Then, Anne the ant walks from the point $(0, 0)$ to the point $(10, 10)$ using a sequence of one-unit right steps and one-unit up steps. Let P be the number of paths Anne could take that pass through both of the points that Eve chose.

The expected value of P is $\binom{20}{10} \cdot \frac{a}{b}$ for relatively prime positive integers a and b. Compute $100a + b$.

Problem 8. For each nonnegative integer n, we define a set H_n of points in the plane as follows:

- H_0 is the unit square $\{(x, y) \mid 0 \leq x, y \leq 1\}$.
- For each $n \geq 1$, we construct H_n from H_{n-1} as follows. Note that H_{n-1} is the union of finitely many square regions $R_1, \ldots, R_k$. For each i, divide R_i into four congruent square quadrants. If n is odd, then the upper-right and lower-left quadrants of each R_i make up H_n. If n is even, then the upper-left and lower-right quadrants of each R_i make up H_n.

Time limit: 40 minutes
Maximum score is 56 points

The figures H_0, H_1, H_2, and H_3 are shown below.

H_0 H_1 H_2 H_3

Suppose that the point $P = (x, y)$ lies in H_n for all $n \geq 0$. The greatest possible value of xy is $\frac{m}{n}$, for relatively prime positive integers m, n. Compute $100m + n$.