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National Internet Math Olympiad

Contest 27

September 26, 2016
8:00 PM – 8:40 PM ET

Calculators and other computational aids are NOT allowed.

Problem 1. Kayla draws three triangles on a sheet of paper. What is the maximum possible number of regions, including the exterior region, that the paper can be divided into by the sides of the triangles?

Problem 2. An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in \overline{AB}$, $Q \in \overline{AC}$, and $N, P \in \overline{BC}$.

Suppose that ABC is an equilateral triangle of side length 2, and that $AMNPQ$ has a line of symmetry perpendicular to BC. Then the area of $AMNPQ$ is $n - p\sqrt{q}$, where n, p, q are positive integers and q is not divisible by the square of a prime. Compute $100n + 10p + q$.

Problem 3. Suppose there exist constants A, B, C, and D such that

$$n^4 = A\binom{n}{4} + B\binom{n}{3} + C\binom{n}{2} + D\binom{n}{1}$$

holds true for all positive integers $n \geq 4$. What is $A + B + C + D$?

Problem 4. Let z be a complex number. If the equation

$$x^3 + (4 - i)x^2 + (2 + 5i)x = z$$

has two roots that form a conjugate pair, find the absolute value of the real part of z.

Problem 5. Find the number of integers n with $1 \leq n \leq 100$ for which $n - \phi(n)$ is prime. Here $\phi(n)$ denotes the number of positive integers less than n which are relatively prime to n.

Problem 6. In $\triangle ABC$, $AB = 4$, $BC = 5$, and $CA = 6$. Circular arcs p, q, r of measure 60° are drawn from A to B, from A to C, and from B to C, respectively, so that p, q lie completely outside $\triangle ABC$ but r does not. Let X, Y, Z be the midpoints of p, q, r, respectively. If $\sin \angle XZY = \dfrac{a\sqrt{b} + c}{d}$, where a, b, c, d are positive integers, $\gcd(a, c, d) = 1$, and b is not divisible by the square of a prime, compute $a + b + c + d$.

Problem 7. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of integers such that $a_1 = 20$, $b_1 = 15$, and for $n \geq 1$,

$$\begin{cases} a_{n+1} = a_n^2 - b_n^2, \\ b_{n+1} = 2a_nb_n - b_n^2 \end{cases}$$

Let $G = a_{10}^2 - a_{10}b_{10} + b_{10}^2$. Determine the number of positive integer factors of G.

Problem 8. Let S be the set of all permutations of $\{1, 2, 3, 4, 5\}$. For $s = (a_1, a_2, a_3, a_4, a_5) \in S$, define $\text{nimo}(s)$ to be the sum of all indices $i \in \{1, 2, 3, 4\}$ for which $a_i > a_{i+1}$. For instance, if $s = (2, 3, 1, 5, 4)$, then $\text{nimo}(s) = 2 + 4 = 6$. Compute

$$\sum_{s \in S} 2^{\text{nimo}(s)}.$$

Time limit: 40 minutes
Maximum score is 56 points