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Revenge 25

April 2016

*April 5, 2016*
*8:00 PM – 8:40 PM ET*

*Calculators and other computational aids are not allowed.*

**Problem 1.** In quadrilateral $ABCD$, $AB \parallel CD$ and $BC \perp AB$. Lines $AC$ and $BD$ intersect at $E$. If $AB = 20$, $BC = 2016$, and $CD = 16$, find the area of $\triangle BCE$.

**Problem 2.** A time is chosen randomly and uniformly in an 24-hour day. The probability that at that time, the (non-reflex) angle between the hour hand and minute hand on a clock is less than $\frac{360}{11}$ degrees is $\frac{m}{n}$ for coprime positive integers $m$ and $n$. Find $100m + n$.

**Problem 3.** David, Kevin, and Michael each choose an integer from the set $\{1, 2, \ldots, 100\}$ randomly, uniformly, and independently of each other. The probability that the positive difference between David's and Kevin's numbers is *strictly* less than that of Kevin's and Michael's numbers is $\frac{m}{n}$, for coprime positive integers $m$ and $n$. Find $100m + n$.

**Problem 4.** Let $S$ be the set of all pairs of positive integers $(x, y)$ for which $2x^2 + 5y^2 \le 5 + 6xy$. Compute $\displaystyle\sum_{(x,y)\in S} (x + y + 100)$.

**Problem 5.** Bob starts with an empty whiteboard. He then repeatedly chooses one of the digits $1, 2, \ldots, 9$ (uniformly at random) and appends it to the end of the currently written number. Bob stops when the number on the board is a multiple of 25. Let $E$ be the expected number of digits that Bob writes. If $E = \frac{m}{n}$ for relatively prime positive integers $m$ and $n$, find $100m + n$.

**Problem 6.** As a reward for working for NIMO, Evan divides 100 indivisible marbles among three of his volunteers: David, Justin, and Michael. (Of course, each volunteer must get at least one marble!) However, Evan knows that, in the middle of the night, Lewis will select a positive integer $n > 1$ and, for each volunteer, steal exactly $\frac{1}{n}$ of his marbles (if possible, i.e. if $n$ divides the number of marbles). In how many ways can Evan distribute the 100 marbles so that Lewis is unable to steal marbles from every volunteer, regardless of which $n$ he selects?

**Problem 7.** Let $(a_1, a_2, \ldots, a_{13})$ be a permutation of $(1, 2, \ldots, 13)$. Ayvak takes this permutation and makes a series of *moves*, each of which consists of choosing an integer $i$ from 1 to 12, inclusive, and swapping the positions of $a_i$ and $a_{i+1}$. Define the *weight* of a permutation to be the minimum number of moves Ayvak needs to turn it into $(1, 2, \ldots, 13)$.

The arithmetic mean of the weights of all permutations $(a_1, \ldots, a_{13})$ of $(1, 2, \ldots, 13)$ for which $a_5 = 9$ is $\frac{m}{n}$, for coprime positive integers $m$ and $n$. Find $100m + n$.

**Problem 8.** Rectangle $EFGH$ with side lengths 8, 9 lies inside rectangle $ABCD$ with side lengths 13, 14, with their corresponding sides parallel. Let $\ell_A, \ell_B, \ell_C, \ell_D$ be the lines connecting $A, B, C, D$, respectively, with the vertex of $EFGH$ closest to them. Let $P = \ell_A \cap \ell_B$, $Q = \ell_B \cap \ell_C$, $R = \ell_C \cap \ell_D$, and $S = \ell_D \cap \ell_A$. Suppose that the greatest possible area of quadrilateral $PQRS$ is $\frac{m}{n}$, for relatively prime positive integers $m$ and $n$. Find $100m + n$.

*Time limit: 40 minutes*
*Maximum score is 56 points*