NIMO

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Contest 24

February 2016

February 22, 2016
8:00 PM – 8:40 PM ET

Calculators and other computational aids are not allowed.

Problem 1. Suppose a_1, a_2, a_3, ... is an arithmetic sequence such that

$$a_1 + a_2 + a_3 + \cdots + a_{48} + a_{49} = 1421.$$

Find the value of $a_1 + a_4 + a_7 + a_{10} + \cdots + a_{49}$.

Problem 2. Michael, David, Evan, Isabella, and Justin compete in the NIMO Super Bowl, a round-robin cereal-eating tournament. Each pair of competitors plays exactly one game, in which each competitor has an equal chance of winning (and there are no ties). The probability that none of the five players wins all of his/her games is $\frac{m}{n}$ for relatively prime positive integers m, n. Compute $100m + n$.

Problem 3. Let f be the quadratic function with leading coefficient 1 whose graph is tangent to that of the lines $y = -5x + 6$ and $y = x - 1$. The sum of the coefficients of f is $\frac{p}{q}$, where p and q are positive relatively prime integers. Find $100p + q$.

Problem 4. Justine has two fair dice, one with sides labeled $1, 2, \ldots, m$ and one with sides labeled $1, 2, \ldots, n$. She rolls both dice once. If $\frac{3}{20}$ is the probability that at least one of the numbers showing is at most 3, find the sum of all distinct possible values of $m + n$.

Problem 5. A wall made of mirrors has the shape of $\triangle ABC$, where $AB = 13$, $BC = 16$, and $CA = 9$. A laser positioned at point A is fired at the midpoint M of BC. The shot reflects about BC and then strikes point P on AB. If $\frac{AM}{MP} = \frac{m}{n}$ for relatively prime positive integers m, n, compute $100m + n$.

Problem 6. Let S be the sum of all positive integers that can be expressed in the form $2^a \cdot 3^b \cdot 5^c$, where a, b, c are positive integers that satisfy $a + b + c = 10$. Find the remainder when S is divided by 1001.

Problem 7. Let A and B be points with $AB = 12$. A point P in the plane of A and B is *special* if there exist points X, Y such that

- P lies on segment XY,
- $PX : PY = 4 : 7$, and
- the circumcircles of AXY and BXY are both tangent to line AB.

A point P that is not special is called *boring*. Compute the smallest integer n such that any two boring points have distance less than $\sqrt{n/10}$ from each other.

Problem 8. For a complex number $z \neq 3,4$, let $F(z)$ denote the real part of $\frac{1}{(3-z)(4-z)}$. If

$$\int_0^1 F\left(\frac{\cos 2\pi t + i \sin 2\pi t}{5}\right) dt = \frac{m}{n}$$

for relatively prime positive integers m and n, find $100m + n$.

Time limit: 40 minutes
Maximum score is 56 points