NIMO

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Contest 23

January 2016

January 6, 2016
8:00 PM – 8:40 PM ET

Calculators and other computational aids are not allowed.

Problem 1. Let m be a positive integer less than 2015. Suppose that the remainder when 2015 is divided by m is n. Compute the largest possible value of n.

Problem 2. For real numbers x and y, define

$$\nabla(x, y) = x - \frac{1}{y}.$$

If

$$\underbrace{\nabla(2, \nabla(2, \nabla(2, \ldots \nabla(2, \nabla(2, 2)) \ldots)))}_{2016\ \nabla\text{s}} = \frac{m}{n}$$

for relatively prime positive integers m, n, compute $100m + n$.

Problem 3. Convex pentagon $ABCDE$ satisfies $AB \parallel DE$, $BE \parallel CD$, $BC \parallel AE$, $AB = 30$, $BC = 18$, $CD = 17$, and $DE = 20$. Find its area.

Problem 4. A fair 100-sided die is rolled twice, giving the numbers a and b in that order. If the probability that $a^2 - 4b$ is a perfect square is $\frac{m}{n}$, where m and n are relatively prime positive integers, compute $100m + n$.

Problem 5. Compute the 100^{th} smallest positive integer n that satisfies the three congruences

$$\left\lfloor \frac{n}{8} \right\rfloor \equiv 3 \pmod 4,$$
$$\left\lfloor \frac{n}{32} \right\rfloor \equiv 2 \pmod 4,$$
$$\left\lfloor \frac{n}{256} \right\rfloor \equiv 1 \pmod 4.$$

Here $\lfloor \cdot \rfloor$ denotes the greatest integer function.

Problem 6. Emma's calculator has ten buttons: one for each digit $1, 2, \ldots, 9$, and one marked "clear". When Emma presses one of the buttons marked with a digit, that digit is appended to the right of the display. When she presses the "clear" button, the display is completely erased. If Emma starts with an empty display and presses five (not necessarily distinct) buttons at random, where all ten buttons have equal probability of being chosen, the expected value of the number produced is $\frac{a}{b}$, for relatively prime positive integers a and b. Find $100a + b$. (Take an empty display to represent the number 0.)

Problem 7. Let $p = 2017$ be a prime. Find the remainder when

$$\left\lfloor \frac{1^p}{p} \right\rfloor + \left\lfloor \frac{2^p}{p} \right\rfloor + \left\lfloor \frac{3^p}{p} \right\rfloor + \cdots + \left\lfloor \frac{2015^p}{p} \right\rfloor$$

is divided by p. Here $\lfloor \cdot \rfloor$ denotes the greatest integer function.

Problem 8. Let $\triangle ABC$ be an equilateral triangle with side length s and P a point in the interior of this triangle. Suppose that PA, PB, and PC are the roots of the polynomial $t^3 - 18t^2 + 91t - 89$. Then s^2 can be written in the form $m + \sqrt{n}$ where m and n are positive integers. Find $100m + n$.

Time limit: 40 minutes
Maximum score is 56 points