NIMO Monthly Contest

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Contest 8 8:00 PM – 8:40 PM EDT

September 24, 2013

1. Let a, b, c, d, e be positive reals satisfying
$$\begin{aligned} a+b &= c \\ a+b+c &= d \\ a+b+c+d &= e. \end{aligned}$$
If $c = 5$, compute $a+b+c+d+e$.

2. A positive integer N has 20 digits when written in base 9 and 13 digits when written in base 27. How many digits does N have when written in base 3?

3. Integers a, b, c are selected independently and at random from the set $\{1, 2, \dots, 10\}$, with replacement. If p is the probability that $a^{b-1}b^{c-1}c^{a-1}$ is a power of two, compute $1000p$.

4. On side $\overline{AB}$ of square $ABCD$, point E is selected. Points F and G are located on sides $\overline{AB}$ and $\overline{AD}$, respectively, such that $\overline{FG} \perp \overline{CE}$. Let P be the intersection point of segments $\overline{FG}$ and $\overline{CE}$. Given that $[EPF] = 1$, $[EPGA] = 8$, and $[CPFB] = 15$, compute $[PGDC]$. (Here $[\mathcal{P}]$ denotes the area of the polygon $\mathcal{P}$.)

5. Let x, y, z be complex numbers satisfying
$$\begin{aligned} z^2 + 5x &= 10z \\ y^2 + 5z &= 10y \\ x^2 + 5y &= 10x \end{aligned}$$
Find the sum of all possible values of z.

6. Let $f(n) = \varphi(n^3)^{-1}$, where $\varphi(n)$ denotes the number of positive integers not greater than n that are relatively prime to n. Suppose
$$\frac{f(1)+f(3)+f(5)+\dots}{f(2)+f(4)+f(6)+\dots} = \frac{m}{n}$$
where m and n are relatively prime positive integers. Compute $100m+n$.

7. Dragon selects three positive real numbers with sum 100, uniformly at random. He asks Cat to copy them down, but Cat gets lazy and rounds them all to the nearest tenth during transcription. If the probability the three new numbers still sum to 100 is $\frac{m}{n}$, where m and n are relatively prime positive integers, compute $100m+n$.

8. The diagonals of convex quadrilateral $BSCT$ meet at the midpoint M of $\overline{ST}$. Lines BT and SC meet at A, and $AB = 91$, $BC = 98$, $CA = 105$. Given that $\overline{AM} \perp \overline{BC}$, find the positive difference between the areas of $\triangle SMC$ and $\triangle BMT$.