NIMO
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Revenge 22

December 2015

December 8, 2015
8:00 PM – 8:40 PM ET

Calculators and other computational aids are not allowed.

Problem 1. Find the value of $\lfloor 1\rfloor + \lfloor 1.7\rfloor + \lfloor 2.4\rfloor + \lfloor 3.1\rfloor + \cdots + \lfloor 99\rfloor$.

Problem 2. Sitting at a desk, Alice writes a nonnegative integer N on a piece of paper, with $N \le 10^{10}$. Interestingly, Celia, sitting opposite Alice at the desk, is able to properly read the number upside-down and gets the same number N, without any leading zeros. (Note that the digits 2, 3, 4, 5, and 7 will not be read properly when turned upside-down.) Find the number of possible values of N.

Problem 3. Right triangle ABC has hypotenuse $AB = 26$, and the inscribed circle of ABC has radius 5. The largest possible value of BC can be expressed as $m + \sqrt{n}$, where m and n are both positive integers. Find $100m + n$.

Problem 4. In rhombus $ABCD$, let M be the midpoint of AB and N be the midpoint of AD. If $CN = 7$ and $DM = 24$, compute AB^2.

Problem 5. In a chemistry experiment, a tube contains 100 particles, 68 on the right and 32 on the left. Each second, if there are a particles on the left side of the tube, some number n of these particles move to the right side, where $n \in \{0, 1, \dots, a\}$ is chosen uniformly at random. In a similar manner, some number of the particles from the right side of the tube move to the left, at the same time. The experiment ends at the moment when all particles are on the same side of the tube. The probability that all particles end on the left side is $\frac{a}{b}$ for relatively prime positive integers a and b. Compute $100a + b$.

Problem 6. Consider a sequence a_0, a_1, …, a_9 of distinct positive integers such that $a_0 = 1$, $a_i < 512$ for all i, and for every $1 \le k \le 9$ there exists $0 \le m \le k-1$ such that

$$(a_k - 2a_m)(a_k - 2a_m - 1) = 0.$$

Let N be the number of these sequences. Find the remainder when N is divided by 1000.

Problem 7. Determine the number of odd integers $1 \le n \le 100$ with the property that

$$\sum_{\substack{1\le k\le n\\ \gcd(k,n)=1}} \cos\left(\frac{2\pi k}{n}\right) = 1 \quad \text{and} \quad \sum_{\substack{1\le k\le n\\ \gcd(k,n)=1}} \sin\left(\frac{2\pi k}{n}\right) = 0.$$

Problem 8. Let $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$ be real numbers which satisfy

$$S_3 = S_{11} = 1, \quad S_7 = S_{15} = -1, \quad \text{and} \quad S_5 = S_9 = S_{13} = 0, \quad \text{where} \quad S_n = \sum_{\substack{1\le i<j\le 8\\ i+j=n}} a_i a_j.$$

(For example, $S_5 = a_1a_4 + a_2a_3$.) Assuming $|a_1| = |a_2| = 1$, the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_8^2$ can be written as $a + \sqrt{b}$ for integers a and b. Compute $a + b$.

Time limit: 40 minutes
Maximum score is 56 points