NIMO
internetolympiad.org
Sponsored by QUIXEY

Contest **21**

October **2015**

October 16, 2015
8:00 PM – 8:40 PM ET

Calculators and other computational aids are not allowed.

Problem 1. In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. A circle of radius r passes through point A and is tangent to line BC at C. If $r = m/n$, where m and n are relatively prime positive integers, find $100m + n$.

Problem 2. In the *Fragmented Game of Spoons*, eight players sit in a row, each with a hand of four cards. Each round, the first player in the row selects the top card from the stack of unplayed cards and either passes it to the second player, which occurs with probability $\frac{1}{2}$, or swaps it with one of the four cards in his hand, each card having an equal chance of being chosen, and passes the new card to the second player. The second player then takes the card from the first player and chooses a card to pass to the third player in the same way. Play continues until the eighth player is passed a card, at which point the card he chooses to pass is removed from the game and the next round begins. To win, a player must hold four cards of the same number, one of each suit.

During a game, David is the eighth player in the row and needs an Ace of Clubs to win. At the start of the round, the dealer picks up a Ace of Clubs from the deck. Suppose that Justin, the fifth player, also has a Ace of Clubs, and that all other Ace of Clubs cards have been removed. The probability that David is passed an Ace of Clubs during the round is $\frac{m}{n}$, where m and n are positive integers with $\gcd(m, n) = 1$. Find $100m + n$.

Problem 3. Find the sum of all positive integers n such that exactly 2% of the numbers in the set $\{1, 2, \ldots, n\}$ are perfect squares.

Problem 4. Let $f(x, y)$ be a function defined for all pairs of nonnegative integers (x, y), such that $f(0, k) = f(k, 0) = 2^k$ and

$$f(a, b) + f(a + 1, b + 1) = f(a + 1, b) + f(a, b + 1)$$

for all nonnegative integers a, b. Determine the number of positive integers $n \leq 2016$ for which there exist two nonnegative integers a, b such that $f(a, b) = n$.

Problem 5. Find the constant k such that the sum of all $x \geq 0$ satisfying $\sqrt{x}(x+12) = 17x - k$ is 256.

Problem 6. Let $ABCD$ be an isosceles trapezoid with $AD \parallel BC$ and $BC > AD$ such that the distance between the incenters of $\triangle ABC$ and $\triangle DBC$ is 16. If the perimeters of $ABCD$ and ABC are 120 and 114 respectively, then the area of $ABCD$ can be written as $m\sqrt{n}$, where m and n are positive integers with n not divisible by the square of any prime. Find $100m + n$.

Problem 7. Given two positive integers m and n, we say that $m \mid\mid n$ if $m \mid n$ and $\gcd(m,\, n/m) = 1$. Compute the smallest integer greater than

$$\sum_{d \mid 2016} \sum_{m \mid\mid d} \frac{1}{m}.$$

Problem 8. Triangle ABC has $AB = 25$, $AC = 29$, and $BC = 36$. Additionally, Ω and ω are the circumcircle and incircle of $\triangle ABC$. Point D is situated on Ω such that AD is a diameter of Ω, and line AD intersects ω in two distinct points X and Y. Compute XY^2.

Time limit: 40 minutes
Maximum score is 56 points