NIMO

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Contest 20

September 2015

September 23, 2015
8:00 PM – 8:40 PM ET

Calculators and other computational aids are not allowed.

Problem 1. Three fair six-sided dice are labeled with the numbers $\{1,2,3,4,5,6\}$, $\{1,2,3,4,5,6\}$, and $\{1,2,3,7,8,9\}$, respectively. All three dice are rolled. The probability that at least two of the dice have the same value is m/n, where m,n are relatively prime positive integers. Find $100m+n$.

Problem 2. Define the *hotel elevator cubic* as the unique cubic polynomial P for which $P(11) = 11$, $P(12) = 12$, $P(13) = 14$, $P(14) = 15$. What is $P(15)$?

Problem 3. How many positive integers divide at least two of the numbers 120, 144, and 180?

Problem 4. Let $f(n) = \frac{n}{3}$ if n is divisible by 3 and $f(n) = 4n - 10$ otherwise. Find the sum of all positive integers c such that $f^5(c) = 2$. (Here $f^5(x)$ means $f(f(f(f(f(x)))))$.)

Problem 5. For positive integers n, let $s(n)$ be the sum of the digits of n. Over all four-digit positive integers n, which value of n maximizes the ratio $\frac{s(n)}{n}$?

Problem 6. Let ABC be a triangle with $AB = 20$, $AC = 34$, and $BC = 42$. Let ω_1 and ω_2 be the semicircles with diameters $\overline{AB}$ and $\overline{AC}$ erected outwards of $\triangle ABC$ and denote by ℓ the common external tangent to ω_1 and ω_2. The line through A perpendicular to $\overline{BC}$ intersects ℓ at X and BC at Y. The length of $\overline{XY}$ can be written in the form $m + \sqrt{n}$ where m and n are positive integers. Find $100m + n$.

Problem 7. Suppose a, b, c, and d are positive real numbers which satisfy the system of equations

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 &= 762, \\ ab + cd &= 260, \\ ac + bd &= 365, \\ ad + bc &= 244. \end{aligned}$$

Compute $abcd$.

Problem 8. Justin the robot is on a mission to rescue abandoned treasure from a minefield. To do this, he must travel from the point $(0,0,0)$ to $(4,4,4)$ in three-dimensional space, only taking one-unit steps in the positive x, y, or z-directions. However, the evil David anticipated Justin's arrival, and so he has surreptitiously placed a mine at the point $(2,2,2)$. If at any point Justin is at most one unit away from this mine (in any direction), the mine detects his presence and explodes, thwarting Justin.

How many paths can Justin take to reach his destination safely?

Time limit: 40 minutes
Maximum score is 56 points