NIMO

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Contest **18**

March **2015**

March 22, 2015
8:00 PM – 8:40 PM ET

Calculators and other computational aids are not allowed.

Problem 1. A function f from the positive integers to the nonnegative integers is defined recursively by $f(1) = 0$ and $f(n+1) = 2^{f(n)}$ for every positive integer n. What is the smallest n such that $f(n)$ exceeds the number of atoms in the observable universe (approximately 10^{80})?

Problem 2. There exists a unique strictly increasing arithmetic sequence $\{a_i\}_{i=1}^{100}$ of positive integers such that

$$a_1 + a_4 + a_9 + \cdots + a_{100} = 1000,$$

where the summation runs over all terms of the form a_{i^2} for $1 \le i \le 10$. Find a_{50}.

Problem 3. Let $ABCD$ be a rectangle with $AB = 6$ and $BC = 6\sqrt{3}$. We construct four semicircles ω_1, ω_2, ω_3, ω_4 whose diameters are the segments AB, BC, CD, DA. It is given that ω_i and ω_{i+1} intersect at some point X_i in the interior of $ABCD$ for every $i = 1, 2, 3, 4$ (indices taken modulo 4). Compute the square of the area of $X_1X_2X_3X_4$.

Problem 4. Find the sum of all positive integers $1 \le k \le 99$ such that there exist positive integers a and b with the property that

$$x^{100} - ax^k + b = (x^2 - 2x + 1)P(x)$$

for some polynomial P with integer coefficients.

Problem 5. Compute the number of subsets S of $\{0, 1, \ldots, 14\}$ with the property that for each $n = 0, 1, \ldots, 6$, either n is in S or both of $2n+1$ and $2n+2$ are in S.

Problem 6. Let ABC be a triangle with $AB = 5$, $BC = 7$, and $CA = 8$. Let D be a point on BC, and define points B' and C' on line AD (or its extension) such that $BB' \perp AD$ and $CC' \perp AD$. If $B'A = B'C'$, then the ratio $BD : DC$ can be expressed in the form $m : n$, where m and n are relatively prime positive integers. Compute $100m + n$.

Problem 7. In a 4×4 grid of unit squares, five squares are chosen at random. The probability that no two chosen squares share a side is $\frac{m}{n}$ for positive relatively prime integers m and n. Find $m + n$.

Problem 8. Let ABC be a non-degenerate triangle with incenter I and circumcircle Γ. Denote by M_a the midpoint of the arc $\widehat{BC}$ of Γ not containing A, and define M_b, M_c similarly. Suppose $\triangle ABC$ has inradius 4 and circumradius 9. Compute the maximum possible value of

$$IM_a^2 + IM_b^2 + IM_c^2.$$

Time limit: 40 minutes
Maximum score is 56 points