NIMO
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Revenge 17
January 2015

January 18, 2015
8:00 PM – 8:40 PM ET

Calculators and other computational aids are not allowed.

Problem 1. Let $2000 < N < 2100$ be an integer. Suppose the last day of year N is a Tuesday while the first day of year $N+2$ is a Friday. The fourth Sunday of year $N+3$ is the mth day of January. What is m?

Problem 2. Let $ABCD$ be a square with side length 100. Denote by M the midpoint of AB. Point P is selected inside the square so that $MP = 50$ and $PC = 100$. Compute AP^2.

Problem 3. How many 5-digit numbers N (in base 10) contain no digits greater than 3 and satisfy the equality $\gcd(N, 15) = \gcd(N, 20) = 1$? (The leading digit of N cannot be zero.)

Problem 4. Determine the number of positive integers $a \le 250$ for which the set $\{a+1, a+2, \dots, a+1000\}$ contains

- Exactly 333 multiples of 3,
- Exactly 142 multiples of 7, and
- Exactly 91 multiples of 11.

Problem 5. Let a, b, c, d, e, and f be real numbers. Define the polynomials

$$P(x) = 2x^4 - 26x^3 + ax^2 + bx + c \quad \text{and} \quad Q(x) = 5x^4 - 80x^3 + dx^2 + ex + f.$$

Let S be the set of all complex numbers which are a root of *either* P or Q (or both). Given that $S = \{1, 2, 3, 4, 5\}$, compute $P(6) \cdot Q(6)$.

Problem 6. Let $\triangle ABC$ be a triangle with $BC = 4, CA = 5, AB = 6$, and let O be the circumcenter of $\triangle ABC$. Let O_b and O_c be the reflections of O about lines CA and AB respectively. Suppose BO_b and CO_c intersect at T, and let M be the midpoint of BC. Given that $MT^2 = \frac{p}{q}$ for some coprime positive integers p and q, find $p + q$.

Problem 7. Find the number of ways a series of $+$ and $-$ signs can be inserted between the numbers $0, 1, 2, \cdots, 12$ such that the value of the resulting expression is divisible by 5.

Problem 8. For an integer $30 \le k \le 70$, let M be the maximum possible value of

$$\frac{A}{\gcd(A, B)} \quad \text{where } A = \binom{100}{k} \text{ and } B = \binom{100}{k+3}.$$

Find the remainder when M is divided by 1000.

Time limit: 40 minutes
Maximum score is 56 points