NIMO
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Contest 16
November 2014

November 19, 2014
8:00 PM – 8:40 PM ET

Calculators and other computational aids are not allowed.

Problem 1. For any interval $\mathcal{A}$ in the real number line not containing zero, define its *reciprocal* to be the set of numbers of the form $\frac{1}{x}$ where x is an element in $\mathcal{A}$. Compute the number of ordered pairs of positive integers (m, n) with $m < n$ such that the length of the interval $[m, n]$ is 10^{10} times the length of its reciprocal.

Problem 2. Let $0^\circ \le \alpha, \beta, \gamma \le 90^\circ$ be angles such that

$$\sin\alpha - \cos\beta = \tan\gamma$$

$$\sin\beta - \cos\alpha = \cot\gamma$$

Compute the sum of all possible values of γ in degrees.

Problem 3. Let $ABCD$ be a square with side length 2. Let M and N be the midpoints of $\overline{BC}$ and $\overline{CD}$ respectively, and let X and Y be the feet of the perpendiculars from A to $\overline{MD}$ and $\overline{NB}$, also respectively. The square of the length of segment $\overline{XY}$ can be written in the form $\frac{p}{q}$ where p and q are positive relatively prime integers. What is $100p + q$?

Problem 4. Let a and b be positive real numbers such that $ab = 2$ and

$$\frac{a}{a+b^2} + \frac{b}{b+a^2} = \frac{7}{8}.$$

Find $a^6 + b^6$.

Problem 5. A positive integer N greater than 1 is described as special if in its base-8 and base-9 representations, both the leading and ending digit of N are equal to 1. What is the smallest special integer in decimal representation?

Problem 6. Bob is making partitions of 10, but he hates even numbers, so he splits 10 up in a special way. He starts with 10, and at each step he takes every even number in the partition and replaces it with a random pair of two smaller positive integers that sum to that even integer. For example, 6 could be replaced with $1+5$, $2+4$, or $3+3$ all with equal probability. He terminates this process when all the numbers in his list are odd. The expected number of integers in his list at the end can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $100m + n$.

Problem 7. Let $\triangle ABC$ have $AB = 6$, $BC = 7$, and $CA = 8$, and denote by ω its circumcircle. Let N be a point on ω such that AN is a diameter of ω. Furthermore, let the tangent to ω at A intersect BC at T, and let the second intersection point of NT with ω be X. The length of $\overline{AX}$ can be written in the form $\frac{m}{\sqrt{n}}$ for positive integers m and n, where n is not divisible by the square of any prime. Find $100m + n$.

Problem 8. Let $p = 2^{16}+1$ be a prime. A sequence of 2^{16} positive integers $\{a_n\}$ is *monotonically bounded* if $1 \le a_i \le i$ for all $1 \le i \le 2^{16}$. We say that a term a_k in the sequence with $2 \le k \le 2^{16} - 1$ is a *mountain* if a_k is greater than both a_{k-1} and a_{k+1}. Evan writes out all possible monotonically bounded sequences. Let N be the total number of mountain terms over all such sequences he writes. Find the remainder when N is divided by p.

Time limit: 40 minutes
Maximum score is 56 points