NIMO

internetolympiad.org

Sponsored by QUIXEY

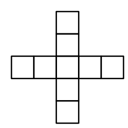
Contest **15**

September **2014**

September 14, 2014
8:00 PM – 8:40 PM ET

Problem 1. Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let D be the point inside triangle ABC with the property that $\overline{BD} \perp \overline{CD}$ and $\overline{AD} \perp \overline{BC}$. Then the length AD can be expressed in the form $m - \sqrt{n}$, where m and n are positive integers. Find $100m + n$.

Problem 2. In the figure below, how many ways are there to select two squares which do not share an edge?



Problem 3. Let $S = \{1, 2, \dots, 2014\}$. Suppose that

$$\sum_{T \subseteq S} i^{|T|} = p + qi$$

where p and q are integers, $i = \sqrt{-1}$, and the summation runs over all 2^{2014} subsets of S. Find the remainder when $|p| + |q|$ is divided by 1000. (Here $|X|$ denotes the number of elements in a set X.)

Problem 4. Points A, B, C, and D lie on a circle such that chords $\overline{AC}$ and $\overline{BD}$ intersect at a point E inside the circle. Suppose that $\angle ADE = \angle CBE = 75^\circ$, $BE = 4$, and $DE = 8$. The value of AB^2 can be written in the form $a + b\sqrt{c}$ for positive integers a, b, and c such that c is not divisible by the square of any prime. Find $a + b + c$.

Problem 5. Let r, s, t be the roots of the polynomial $x^3 + 2x^2 + x - 7$. Then

$$\left(1 + \frac{1}{(r+2)^2}\right)\left(1 + \frac{1}{(s+2)^2}\right)\left(1 + \frac{1}{(t+2)^2}\right) = \frac{m}{n}$$

for relatively prime positive integers m and n. Compute $100m + n$.

Problem 6. For all positive integers k, define $f(k) = k^2 + k + 1$. Compute the largest positive integer n such that

$$2015 f(1^2) f(2^2) \cdots f(n^2) \geq \Big(f(1) f(2) \cdots f(n)\Big)^2.$$

Problem 7. Find the sum of the prime factors of 67208001, given that 23 is one.

Problem 8. For positive integers a, b, and c, define

$$f(a,b,c) = \frac{abc}{\gcd(a,b,c) \cdot \operatorname{lcm}(a,b,c)}.$$

We say that a positive integer n is $f@$ if there exist pairwise distinct positive integers $x, y, z \leq 60$ that satisfy $f(x,y,z) = n$. How many $f@$ integers are there?

Time limit: 40 minutes
Maximum score is 56 points