The Online Math Open Winter Contest
Official Solutions
January 4, 2013–January 15, 2013

Acknowledgments

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Contest Information

Format

The test will start Friday, January 4 and end Tuesday, January 15 (ignore the previous deadline of Monday, January 14). You will have until 7pm EST on January 15 to submit your answers; see here for submission procedures. The test consists of 50 short answer questions, each of which has a nonnegative integer answer no greater than $2^{63} - 2 = 9223372036854775806$. The problem difficulties range from those of AMC problems to those of Olympiad problems. Problems are ordered in roughly increasing order of difficulty.

Team Guidelines

Students may compete in teams of up to four people. Participating students must not have graduated from high school. International students may participate. No student can be a part of more than one team. The members of each team do not get individual accounts; they will all share the team account.

Each team will submit its final answers through its team account. Though teams can save drafts for their answers, the current interface does not allow for much flexibility in communication between team members. We recommend using Google Docs and Spreadsheets to discuss problems and compare answers, especially if teammates cannot communicate in person. Teams may spend as much time as they like on the test before the deadline.

Aids

Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids are not allowed. This is includes (but is not limited to) Geogebra and graphing calculators. **Published print and electronic resources are not permitted.** (This is a change from last year's rules.)

Four-function calculators are permitted on the Online Math Open. That is, calculators which perform only the four basic arithmetic operations (+-*/) may be used. Any other computational aids such as scientific and graphing calculators, computer programs and applications such as Mathematica, and online databases are prohibited. All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.

Clarifications

Clarifications will be posted as they are answered. For the Winter 2012-2013 Contest, they will be posted here. If you have a question about a problem, please email OnlineMathOpenTeam@gmail.com with "Clarification" in the subject. We have the right to deny clarification requests that we feel we cannot answer.

Scoring

Each problem will be worth one point. Ties will be broken based on the "hardest" problem that a team answered correctly. Remaining ties will be broken by the second hardest problem solved, and so on. Problem X is defined to be "harder" than Problem Y if and only if

(i) X was solved by less teams than Y, OR

(ii) X and Y were solved by the same number of teams and X appeared later in the test than Y.

Note: This is a change from prior tiebreaking systems. However, we will still order the problems by approximate difficulty.

Results

After the contest is over, we will release the answers to the problems within the next day. If you have a protest about an answer, you may send an email to OnlineMathOpenTeam@gmail.com (Include "Protest" in the subject). Solutions and results will be released in the following weeks.

1. Let x be the answer to this problem. For what real number a is the answer to this problem also $a - x$?

   Answer. $\boxed{0}$.

   Solution. $a = x$, so $a - x = 0$ must be the answer. $\square$

   This problem was proposed by Ray Li.

2. The number 123454321 is written on a blackboard. Evan walks by and erases some (but not all) of the digits, and notices that the resulting number (when spaces are removed) is divisible by 9. What is the fewest number of digits he could have erased?

   Answer. $\boxed{2}$.

   Solution. The sum of the digits needs to be divisible by 9, so we need to remove 7 at minimum. Since there is no 7, we can remove a 2 and a 5 or a 3 and a 4: 2 digits. $\square$

   This problem was proposed by Ray Li. This solution was given by ahaanomegas on AoPS.

3. Three lines m, n, and ℓ lie in a plane such that no two are parallel. Lines m and n meet at an acute angle of 14°, and lines m and ℓ meet at an acute angle of 20°. Find, in degrees, the sum of all possible acute angles formed by lines n and ℓ.

   Answer. $\boxed{40}$.

   Solution. There are two possible angles: $20^\circ - 16^\circ$ and $20^\circ + 16^\circ$ (note that both are acute), so the sum is $2 \cdot 20^\circ = 40^\circ$. $\square$

   This problem was proposed by Ray Li.

4. For how many ordered pairs of positive integers (a, b) with $a, b < 1000$ is it true that a times b is equal to b^2 divided by a? For example, 3 times 9 is equal to 9^2 divided by 3.

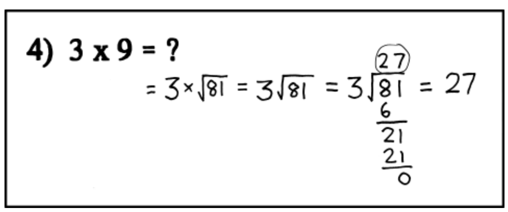

Figure 1: xkcd 759

   Answer. $\boxed{31}$.

   Solution. $ab = \frac{b^2}{a}$ is equivalent to $a^2 = b$, so we simply want the number of perfect squares less than 1000, or 31. $\square$

   This problem was proposed by Ray Li.

5. At the Mountain School, Micchell is assigned a *submissiveness rating* of 3.0 or 4.0 for each class he takes. His *college potential* is then defined as the average of his submissiveness ratings over all classes taken. After taking 40 classes, Micchell has a college potential of 3.975. Unfortunately, he needs a college potential of at least 3.995 to get into the South Harmon Institute of Technology. Otherwise, he becomes a rock. Assuming he receives a submissiveness rating of 4.0 in every class he takes from now on, how many more classes does he need to take in order to get into the South Harmon Institute of Technology?

   Answer. $\boxed{160}$.

Solution. If at any point Micchell has a and b ratings of 4.0 and 3.0, respectively, his college potential is $\frac{4a+3b}{a+b} = 4 - \frac{b}{a+b}$. Thus after taking 40 classes, he has received exactly one submissiveness rating of 3.0. In order to get his college potential up to at least 3.995, he needs $4 - \frac{1}{x} \geq 3.995$, where x is the number of total classes taken (including the original 40); solving, we get $x \geq 200$. Since $x = 200$ also suffices, Micchell needs to take at least $200 - 40 = 160$ more classes.

In other words, he should clearly just try to be a rock. □

This problem was proposed by Victor Wang.

6. Circle S_1 has radius 5. Circle S_2 has radius 7 and has its center lying on S_1. Circle S_3 has an integer radius and has its center lying on S_2. If the center of S_1 lies on S_3, how many possible values are there for the radius of S_3?

   Answer. $\boxed{11}$.

   Solution. By the triangle inequality, the possible radii are those between $7 - 5$ and $7 + 5$, giving us 11 possible values. □

   This problem was proposed by Ray Li.

7. Jacob's analog clock has 12 equally spaced tick marks on the perimeter, but all the digits have been erased, so he doesn't know which tick mark corresponds to which hour. Jacob takes an arbitrary tick mark and measures clockwise to the hour hand and minute hand. He measures that the minute hand is 300 degrees clockwise of the tick mark, and that the hour hand is 70 degrees clockwise of the same tick mark. If it is currently morning, how many minutes past midnight is it?

   Answer. $\boxed{500}$.

   Solution. The hour's degree from the closest tick mark is 10 degrees clockwise, which means 20 minutes. This is 120 degrees clockwise from tick mark 12. So Jacob chooses tick mark 6, and the current time is 8:20, so 500 minutes. □

   This problem was proposed by Ray Li. This solution was given by chaotic_iak on AoPS.

8. How many ways are there to choose (not necessarily distinct) integers a, b, c from the set $\{1, 2, 3, 4\}$ such that $a^{(b^c)}$ is divisible by 4?

   Answer. $\boxed{28}$.

   Solution. Basic casework. Note that, if $a = 1$ or $a = 3$, the power is going to be odd, so no change of 4-divisibility. The interesting part is $a = 2$ and $a = 4$.

   If $a = 2$, as long as $b^c \geq 2$, we're good. If $b = 1$, $b^c = 1$, so that fails. If $b = 2$, $b = 3$, or $b = 4$, we have 4 choices of c for each, making 12.

   If $a = 4$, all combinations of b and c work. There are $4^2 = 16$ ways we can do this.

   Thus, the answer is $12 + 16 = 28$. □

   This problem was proposed by Ray Li. This solution was given by ahaanomegas on AoPS.

9. David has a collection of 40 rocks, 30 stones, 20 minerals and 10 gemstones. An operation consists of removing three objects, no two of the same type. What is the maximum number of operations he can possibly perform?

   Answer. $\boxed{30}$.

Solution. Each move uses at least one mineral or gemstone, so we cannot do better than 30. (Alternatively, as mcdonalds106_7 notes on AoPS, we can note that each move uses at least two stones, minerals, or gemstones.) On the other hand, this is easily achieved by taking a rock and a stone (and then either a mineral or a gemstone) each time. $\square$

This problem was proposed by Ray Li.

10. At certain store, a package of 3 apples and 12 oranges costs 5 dollars, and a package of 20 apples and 5 oranges costs 13 dollars. Given that apples and oranges can only be bought in these two packages, what is the minimum nonzero amount of dollars that must be spent to have an equal number of apples and oranges?

Answer. $\boxed{64}$.

Solution. The (signed) difference between apples and oranges (aka apples - oranges) must be 0. The first package gives a difference -9 while the second gives $+15$, so we need to have 5 of the first packages and 3 of the second packages for 64 dollars. (Just to check, there are 75 apples and 75 oranges now.) $\square$

This problem was proposed by Ray Li. This solution was given by chaotic_iak on AoPS.

11. Let A, B, and C be distinct points on a line with $AB = AC = 1$. Square $ABDE$ and equilateral triangle ACF are drawn on the same side of line BC. What is the degree measure of the acute angle formed by lines EC and BF?

Answer. $\boxed{75}$.

Solution. The desired angle is just $\angle FBC + \angle ECB = \frac{180^\circ - 120^\circ}{2} + 45^\circ = 75^\circ$, where we use the fact that $AB = AF$. $\square$

This problem was proposed by Ray Li.

12. There are 25 ants on a number line; five at each of the coordinates 1, 2, 3, 4, and 5. Each minute, one ant moves from its current position to a position one unit away. What is the minimum number of minutes that must pass before it is possible for no two ants to be on the same coordinate?

Answer. $\boxed{126}$.

Solution. Consider the sum of the (absolute) distances from the original centroid (3) of the ants to each of the ants. This quantity changes by at most 1 each time, so the answer is at least $(0 + 1 + 1 + \cdots + 12 + 12) - 5(0 + 1 + 1 + 2 + 2) = 126$, which is clearly achievable (as long as we don't go beyond 3 ± 12). $\square$

This problem was proposed by Ray Li.

13. There are three flies of negligible size that start at the same position on a circular track with circumference 1000 meters. They fly clockwise at speeds of 2, 6, and k meters per second, respectively, where k is some positive integer with $7 \le k \le 2013$. Suppose that at some point in time, all three flies meet at a location different from their starting point. How many possible values of k are there?

Answer. $\boxed{501}$.

Solution. Working modulo 1000 (allowing fractional parts, as we do mod 1) tells us that precisely $k \equiv 2 \pmod 4$ work. $\square$

This problem was proposed by Ray Li.

14. What is the smallest perfect square larger than 1 with a perfect square number of positive integer factors?

Answer. $\boxed{36}$.

Solution. Clearly our number will either be of the form p^8 or p^2q^2, so our answer is $6^2 = 36$. (Anything with at least three distinct prime factors will be at least 30^2.) ☐

This problem was proposed by Ray Li.

15. A permutation $a_1, a_2, \ldots, a_{13}$ of the numbers from 1 to 13 is given such that $a_i > 5$ for $i = 1, 2, 3, 4, 5$. Determine the maximum possible value of

$$a_{a_1} + a_{a_2} + a_{a_3} + a_{a_4} + a_{a_5}.$$

Answer. $\boxed{45}$.

Solution. a_1 through a_5 take up 5 numbers greater than 5, so we can't do better than $13 + 12 + 11 + 5 + 4 = 45$. This is achieved whenever $\{a_1, \ldots, a_5\} = \{6, 7, 8, 9, 10\}$ and $\{a_6, a_7, a_8, a_9, a_{10}\} = \{4, 5, 11, 12, 13\}$. ☐

This problem was proposed by Evan Chen.

16. Let S_1 and S_2 be two circles intersecting at points A and B. Let C and D be points on S_1 and S_2 respectively such that line CD is tangent to both circles and A is closer to line CD than B. If $\angle BCA = 52^\circ$ and $\angle BDA = 32^\circ$, determine the degree measure of $\angle CBD$.

Answer. $\boxed{48}$.

Solution. Note that $\angle CBA = \angle DCA, \angle DBA = \angle CDA$. Let $\angle CBA = x, \angle DBA = y$; then $\angle CBD = x + y$, $\angle BCD = 52 + x$, $\angle CDB = 32 + y$. Therefore, $2(x + y) + 52 + 32 = 180$ yields $\angle CBD = \frac{180-52-32}{2} = 48^\circ$. ☐

This problem was proposed by Ray Li. This solution was given by thugzmath10 on AoPS.

17. Determine the number of ordered pairs of positive integers (x, y) with $y < x \le 100$ such that $x^2 - y^2$ and $x^3 - y^3$ are relatively prime. (Two numbers are *relatively prime* if they have no common factor other than 1.)

Answer. $\boxed{99}$.

Solution. Clearly we need $x - y = 1$. It's easy to check that $\gcd(x + y, x^2 + xy + y^2) = 1$ for all such x, y, so the answer is 99. ☐

This problem was proposed by Ray Li.

18. Determine the absolute value of the sum

$$\lfloor 2013 \sin 0^\circ \rfloor + \lfloor 2013 \sin 1^\circ \rfloor + \cdots + \lfloor 2013 \sin 359^\circ \rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

(You may use the fact that $\sin n^\circ$ is irrational for positive integers n not divisible by 30.)

Answer. $\boxed{178}$.

Solution. $\lfloor x \rfloor + \lfloor -x \rfloor = -1$ when x is not an integer, so from $\sin(x + \pi) = -\sin(x)$ we get ans answer is $-178 + \lfloor 1 \rfloor + \lfloor -1 \rfloor = 0$. ☐

This problem was proposed by Ray Li.

19. A, B, C are points in the plane such that $\angle ABC = 90^\circ$. Circles with diameters BA and BC meet at D. If $BA = 20$ and $BC = 21$, then the length of segment BD can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. What is $m + n$?

Answer. $\boxed{449}$.

Solution. Let the circle with diameter BA intersect AC at E. Similarly, let the circle with diameter BC intersect AC at F. We easily find $\angle BDA = 90$ and $\angle BFC = 90$, so E and F are the same point, the intersection of AC and the altitude from B. The answer follows. $\square$

This problem was proposed by Ray Li. This solution was given by BOGTRO on AoPS.

20. Let $a_1, a_2, \ldots, a_{2013}$ be a permutation of the numbers from 1 to 2013. Let $A_n = \frac{a_1+a_2+\cdots+a_n}{n}$ for $n = 1, 2, \ldots, 2013$. If the smallest possible difference between the largest and smallest values of $A_1, A_2, \ldots, A_{2013}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

Answer. $\boxed{3}$.

Solution. Looking at $|A_1 - A_2|$ we see the answer is at least $1/2$. Equality can be achieved by $1007, 1008, 1006, 1009, 1005, \ldots$. $\square$

This problem was proposed by Ray Li.

21. Dirock has a very neat rectangular backyard that can be represented as a 32×32 grid of unit squares. The rows and columns are each numbered $1, 2, \ldots, 32$. Dirock is very fond of rocks, and places a rock in every grid square whose row and column number are both divisible by 3. Dirock would like to build a rectangular fence with vertices at the centers of grid squares and sides parallel to the sides of the yard such that

    a) The fence does not pass through any grid squares containing rocks;
    b) The interior of the fence contains exactly 5 rocks.

    In how many ways can this be done?

Answer. $\boxed{1920}$.

Solution. 5 rocks? In a rectangle? Oh no I guess the only way to do that is have 5 rocks in a row. What a rocky start to a rocky problem. Or maybe it's not that unfortunate.

For each row (with rocks), there are 10 rocks so there are 6 sequences of 5 consecutive rocks. There are 10 rows, and also we can have both vertical and horizontal groups of rocks, so we have the $6 \times 10 \times 2 = 120$ groups of 5 abiotic likenesses of Geodudes (substitute Gravelers and Golems as necessary).

Let's construct a frame for each of our 5-rock gardens. There are 2 possible top boundaries, 2 possible lower boundaries, 2 possible left boundaries, and 2 possible right boundaries, so there are $2^4 = 16$ different frames we can choose from. And what do you know, everything is *conveniently* symmetric: it's the same for every group of rocks. WOW WHAT A COINCIDENCE I guess we must be lucky to get this problem huh

So by the multiplication principle (which everyone knows but like no one actually knows by name), our answer is $120 \times 16 = \boxed{1920}$ = year of ratification of 19^{th} amendment so clearly the test writers wanted to make a point about women's suffrage. $\square$

This problem was proposed by Ray Li. This solution was given by Past on AoPS.

22. In triangle ABC, $AB = 28$, $AC = 36$, and $BC = 32$. Let D be the point on segment BC satisfying $\angle BAD = \angle DAC$, and let E be the unique point such that $DE \parallel AB$ and line AE is tangent to the circumcircle of ABC. Find the length of segment AE.

Answer. $\boxed{18}$.

Solution. We have $BD = \frac{28\cdot 32}{28+36} = 14$ and $CD = 32-14 = 18$. Since $DE \parallel AB$, $\angle CDE = \angle CBA$ and $\angle BAD = \angle EDA$. But $\angle ABC = \angle CAE$, so it follows that $\angle CDE = \angle CBA$ (which implies $AECD$ is cyclic), $\angle CAD = \angle EDA$ and $\angle CDA = \angle EAD$. As AD is the common segment of triangles CAD and EDA, by ASA Congurence Postulate we have $\triangle CAD \cong \triangle EDA$. Therefore, $AE = CD = 18$. □

This problem was proposed by Ray Li. This solution was given by thugzmath10 on AoPS.

23. A set of 10 distinct integers S is chosen. Let M be the number of nonempty subsets of S whose elements have an even sum. What is the minimum possible value of M?

Answer. $\boxed{511}$.

Clarifications.

- S is the "set of 10 distinct integers" from the first sentence.

Solution. If all the elements are even we get $2^{10}-1$. If at least one is odd then we get at least $\frac{1}{2}2^{10}-1$, which is achieved by all 1s, for instance. □

This problem was proposed by Ray Li.

24. For a permutation π of the integers from 1 to 10, define

$$S(\pi) = \sum_{i=1}^{9} (\pi(i) - \pi(i+1)) \cdot (4 + \pi(i) + \pi(i+1)),$$

where $\pi(i)$ denotes the ith element of the permutation. Suppose that M is the maximum possible value of $S(\pi)$ over all permutations π of the integers from 1 to 10. Determine the number of permutations π for which $S(\pi) = M$.

Answer. $\boxed{40320}$.

Solution. The sum telescopes, giving us an answer of $8! = 40320$. □

This problem was proposed by Ray Li.

25. Positive integers $x, y, z \le 100$ satisfy

$$\begin{aligned} 1099x + 901y + 1110z &= 59800 \\ 109x + 991y + 101z &= 44556 \end{aligned}$$

Compute $10000x + 100y + z$.

Answer. $\boxed{34316}$.

Solution. Subtracting the first equation from 11 times the second, we obtain $100x + 10000y + z = 430316$. Since $x, y, z \le 100$ this implies $(x, y, z) = (3, 43, 16)$, so $10000x + 100y + z = 34316$. □

Solution 2. Taking the equations modulo 100, we obtain $-x + y + 10z \equiv 0 \pmod{100}$ and $9x - 9y + z \equiv 56 \pmod{100}$, whence $91z \equiv 56 \pmod{100}$. But $9 \cdot 11 = 99 \equiv -1 \pmod{100}$ means $z \equiv 11 \cdot 91z \equiv 11 \cdot 56 \equiv 16 \pmod{100}$, so $z = 16$.

On the other hand, subtracting the former from 10 times the latter gives $-9x + 9009y - 100z = 385760$, so

$$385760 + 1600 + 9 \le 9009y \le 385760 + 1600 + 900 \implies y = 43,$$

since $1 \le x, y, z \le 100$ are positive integers (observe that $9009 \cdot 43 = 387387$). Finally, $x \equiv y + 10z \equiv 43 + 60 \equiv 3 \pmod{100}$, so $(x, y, z) = (3, 43, 16)$ and $10000x + 100y + z = 34316$. □

This problem was proposed by Evan Chen. The second solution was given by Victor Wang.

26. In triangle ABC, F is on segment AB such that CF bisects $\angle ACB$. Points D and E are on line CF such that lines AD, BE are perpendicular to CF. M is the midpoint of AB. If $ME = 13$, $AD = 15$, and $BE = 25$, find $AC + CB$.

Answer. $\boxed{104}$.

Solution. Note that $\triangle BEF$ is similar to $\triangle ADF$, with scale factor 5 to 3. If we let $EF = 5a$, then $DF = 3a$. Also note that $\triangle ADC$ is similar to $\triangle BEC$, with scale factor 5 to 3. Since $DE = 8a$, $CD = 12a$. By Pythagorean theorem, $AF = 3\sqrt{a^2+25}$ and $BF = 5\sqrt{a^2+25}$. Also by Pythagorean theorem, $AE = \sqrt{64a^2+125}$.

By Stewart's theorem on triangle AEB using ME, AM, and BM as sides in the calculation, we have (letting $AM = BM = x$)

$$625x + (64a^2 + 225)x = 2x \cdot x \cdot x + 2x \cdot 169,$$

so $850 + 64a^2 = 2x^2 + 338$ and $x = 4\sqrt{2a^2+16}$. So $2x = AB = 8\sqrt{2a^2+16}$ but this also equals $AF + BF$ or $8\sqrt{a^2+25}$ so equating them we get $a = 3$ and $AC = 39$, $BC = 65$ so the answer is then 104. $\square$

This problem was proposed by Ray Li.

27. Geodude wants to assign one of the integers $1, 2, 3, \dots, 11$ to each lattice point (x, y, z) in a 3D Cartesian coordinate system. In how many ways can Geodude do this if for every lattice parallelogram $ABCD$, the positive difference between the sum of the numbers assigned to A and C and the sum of the numbers assigned to B and D must be a multiple of 11? (A *lattice point* is a point with all integer coordinates. A *lattice parallelogram* is a parallelogram with all four vertices lying on lattice points. Here, we say four not necessarily distinct points A, B, C, D form a *parallelogram* $ABCD$ if and only if the midpoint of segment AC coincides with the midpoint of segment BD.)

Answer. $\boxed{14641}$.

Clarifications.

- The "positive difference" between two real numbers x and y is the quantity $|x - y|$. Note that this may be zero.
- The last sentence was added to remove confusion about "degenerate parallelograms."

Solution. Let $f(x, y, z)$ be the number written at (x, y, z), and $g(x, y, z) = f(x, y, z) - f(0, 0, 0)$. Then considering all parallelograms $ABCD$ with $A = (0, 0, 0)$, we get

$$11 \mid g(x_1 + x_2, y_1 + y_2, z_1 + z_2) - g(x_1, y_1, z_1) - g(x_2, y_2, z_2)$$

for all integers $x_1, y_1, \dots, z_2$. But then it's not hard to prove

$$g(x, y, z) \equiv xg(1, 0, 0) + yg(0, 1, 0) + zg(0, 0, 1) \pmod{11}$$

for all integers x, y, z. (Try it!)

On the other hand, any choice of $f(0, 0, 0)$, $g(1, 0, 0)$, $g(0, 1, 0)$, and $g(0, 0, 1)$ yields a distinct working assignment f (Why?), so the desired answer is $11^4 = 14641$. $\square$

This problem was proposed by Victor Wang.

28. Let S be the set of all lattice points (x, y) in the plane satisfying $|x| + |y| \le 10$. Let $P_1, P_2, \ldots, P_{2013}$ be a sequence of 2013 (not necessarily distinct) points such that for every point Q in S, there exists at least one index i such that $1 \le i \le 2013$ and $P_i = Q$. Suppose that the minimum possible value of $|P_1P_2| + |P_2P_3| + \cdots + |P_{2012}P_{2013}|$ can be expressed in the form $a + b\sqrt{c}$, where a, b, c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$. (A *lattice point* is a point with all integer coordinates.)

    Answer. $\boxed{222}$.

    Clarifications.

    - $k = 2013$, i.e. the problem should read, "…there exists at least one index i such that $1 \le i \le 2013$ …". An earlier version of the test read $1 \le i \le k$.

    Solution. More generally, let S_n be the set of all lattice points (x, y) such that $|x| + |y| \le n$. Color (x, y) black if $x + y - n$ is even and white otherwise. Notice that there are $(n + 1)^2$ black points and n^2 white points, so there are at least $2n$ segments in this broken line that go from a black point to a black point, and the length of each such segment is at least $\sqrt{2}$. There are $2n^2 + 2n$ total segments, so the length of the broken line is at least $2n^2 + 2n\sqrt{2}$. The construction is not hard to find (basically a spiral). □

    This problem was proposed by Anderson Wang.

29. Let $\phi(n)$ denote the number of positive integers less than or equal to n that are relatively prime to n, and let $d(n)$ denote the number of positive integer divisors of n. For example, $\phi(6) = 2$ and $d(6) = 4$. Find the sum of all odd integers $n \le 5000$ such that $n \mid \phi(n)d(n)$.

    Answer. $\boxed{2903}$.

    Solution. Note $3^8 > 5000$, so all primes dividing n are at most 8 and thus 3, 5, or 7. Doing casework on the set of primes (hardest part is $3^6 \cdot 7 = 5103 > 5000$) we get $1 + 3^2 + 3^5 + 5^4 + 3^4 \cdot 5^2$. □

    This problem was proposed by Alex Zhu.

30. Pairwise distinct points $P_1, P_2, \ldots, P_{16}$ lie on the perimeter of a square with side length 4 centered at O such that $|P_iP_{i+1}| = 1$ for $i = 1, 2, \ldots, 16$. (We take P_{17} to be the point P_1.) We construct points $Q_1, Q_2, \ldots, Q_{16}$ as follows: for each i, a fair coin is flipped. If it lands heads, we define Q_i to be P_i; otherwise, we define Q_i to be the reflection of P_i over O. (So, it is possible for some of the Q_i to coincide.) Let D be the length of the vector $\overrightarrow{OQ_1} + \overrightarrow{OQ_2} + \cdots + \overrightarrow{OQ_{16}}$. Compute the expected value of D^2.

    Answer. $\boxed{88}$.

    Solution. What's $\sum(\pm a \pm b \pm c)^2$ over all $2^3 = 8$ choices of signs depends only on $a^2 + b^2 + c^2$. This generalizes! □

    This problem was proposed by Ray Li.

31. Beyond the Point of No Return is a large lake containing 2013 islands arranged at the vertices of a regular 2013-gon. Adjacent islands are joined with exactly two bridges. Christine starts on one of the islands with the intention of burning all the bridges. Each minute, if the island she is on has at least one bridge still joined to it, she randomly selects one such bridge, crosses it, and immediately burns it. Otherwise, she stops.

    If the probability Christine burns all the bridges before she stops can be written as $\frac{m}{n}$ for relatively prime positive integers m and n, find the remainder when $m + n$ is divided by 1000.

    Answer. $\boxed{113}$.

Solution. We either cycle through the whole 2013-gon first (after which Christine will automatically succeed) or "retrace" first right after passing through $k \in [1, 2012]$ bridges. In the first case we have a probability of

$$(4/4)(2/3)^{2012}(2/2)(1/1)^{2012} = (2/3)^{2012};$$

in the second we have

$$(4/4)(2/3)^{k-1}(1/3)(1/1)^{k-1}(2/2)(2/3)^{2012-k}(1/1)^{2013-k} = (1/3)(2/3)^{2011}.$$

Summing up we get $1007(2/3)^{2012}$ so $m = 1007 \cdot 2^{2012}$ and $n = 3^{2012}$ (since $3 \nmid 1007$). Our answer is thus

$$1007 \cdot 2^{2012} + 3^{2012} \equiv 7 \cdot 2^{12} + 3^{12} \equiv 113 \pmod{1000}.$$

(To speed up calculations, we can work mod 8 and mod 125 first and then use CRT.) □

This problem was proposed by Evan Chen.

32. In $\triangle ABC$ with incenter I, $AB = 61$, $AC = 51$, and $BC = 71$. The circumcircles of triangles AIB and AIC meet line BC at points D ($D \neq B$) and E ($E \neq C$), respectively. Determine the length of segment DE.

Answer. $\boxed{41}$.

Solution. Angle chasing gives $\triangle IDC \cong \triangle IAC$, so $CA = CD$ and similarly, $BA = BE$. Thus $DE = BE + CD - BC = AB + AC - BC = 41$. □

Comment. I is the circumcenter of $\triangle ADE$, and D, E always lie on the same side of line AI as B, C, respectively. This gives us another way to define D, E.

This problem was proposed by James Tao.

33. Let n be a positive integer. E. Chen and E. Chen play a game on the n^2 points of an $n \times n$ lattice grid. They alternately mark points on the grid such that no player marks a point that is on or inside a non-degenerate triangle formed by three marked points. Each point can be marked only once. The game ends when no player can make a move, and the last player to make a move wins. Determine the number of values of n between 1 and 2013 (inclusive) for which the first player can guarantee a win, regardless of the moves that the second player makes.

Answer. $\boxed{1007}$.

Solution. If n is odd, the first player should take the center square. Then for the second player's move, the first player should choose the reflection across the center square. It is clear that this will not be in the convex hull of the chosen points. So here the first player always wins.

If n is even, then the second player should take the reflection across the center square. So the second player always win in this case.

The answer is thus 1007. □

This problem was proposed by Ray Li. This solution was given by Yang Liu.

34. For positive integers n, let $s(n)$ denote the sum of the squares of the positive integers less than or equal to n that are relatively prime to n. Find the greatest integer less than or equal to

$$\sum_{n \mid 2013} \frac{s(n)}{n^2},$$

where the summation runs over all positive integers n dividing 2013.

Answer. $\boxed{671}$.

Solution. We have

$$\sum_{n|N} \frac{s(n)}{n^2} = \frac{1^2 + 2^2 + \cdots + N^2}{N^2} = \frac{(N+1)(2N+1)}{6N}.$$

□

Solution 2. For $d \mid n$, let $f(n, d)$ equal $\sum_{k=1}^{n/d} (dk)^2 = \frac{n(n+d)(2n+d)}{6d}$ if d is square-free and 0 otherwise. Then by PIE, we have

$$\begin{aligned} s(n) &= \sum_{d|n} (-1)^{\omega(d)} f(n, d) \\ &= \frac{n}{6}\left(\prod_{p|n}(1-p) + 3n(0) + 2n^2 \prod_{p|n}\left(1 - \frac{1}{p}\right)\right) \\ &= \frac{n}{6}\prod_{p|n}(1-p) + \frac{n^2}{3}\phi(n) \end{aligned}$$

for $n > 1$ (if $n = 1$ the $3n$ term doesn't cancel out, so we instead have $s(1) = 1$), where $\omega(m)$ denotes the number of distinct prime factors of m. It follows that

$$\begin{aligned} \sum_{d|n} \frac{s(d)}{d^2} &= 1 + \frac{1}{3}\sum_{1<d|n} \phi(d) + \frac{1}{6}\sum_{1<d|n} \frac{1}{d}\prod_{p|d}(1-p) \\ &= 1 + \frac{1}{3}(n - \phi(1)) + \frac{1}{6}\left(-1 + \prod_{p|n}\left(1 + \frac{1-p}{p} + \cdots + \frac{1-p}{p^{v_p(n)}}\right)\right) \\ &= 1 + \frac{n-1}{3} + \frac{1}{6}\left(-1 + \prod_{p|n} \frac{1}{p^{v_p(n)}}\right) \\ &= \frac{n+2}{3} - \frac{1}{6} + \frac{1}{6n} \\ &= \frac{2n^2 + 3n + 1}{6n} \\ &= \frac{(n+1)(2n+1)}{6n} \\ &= \frac{n}{3} + \frac{1}{2} + \frac{1}{6n}. \end{aligned}$$

Since $3 \mid 2013$, the floor of this quantity is $\frac{2013}{3} = 671$. □

This problem was proposed by Ray Li. The second solution was found by a rock.

35. The rows and columns of a 7×7 grid are each numbered $1, 2, \ldots, 7$. In how many ways can one choose 8 cells of this grid such that for every two chosen cells X and Y, either the positive difference of their row numbers is at least 3, or the positive difference of their column numbers is at least 3?

Answer. $\boxed{51}$.

Clarifications.

- The "or" here is inclusive (as by convention, despite the "either"), i.e. X and Y are permitted if and only if they satisfy the row condition, the column condition, or both.

Solution. This bijects to the number of ways of placing 8 nonintersecting 3×3 squares in a 9×9 grid.

3-color the columns $A, B, C, A, \ldots$. Label the 9×9 grid with (i,j) (row i, column j). If we don't have $(i,j)\equiv(2,2) \pmod 3$, we can get a lot of restrictions. Identify squares by centers. For stuff symmetric (under rotation or reflection) to $(3,3),(3,4),(4,3),(4,4)$ we can't have any squares (or else we kill at least 4 A's; for the symmetric squares we just rotate/reflect the 3-coloring as needed).

We now show that in every working configuration, at least two rows or two columns of $\{2,5,8\}\times\{2,5,8\}$ are filled out* (and these all trivially work, so it reduces to PIE). If all squares are $(2,2) \pmod 3$ this is obvious. Otherwise up to rotation/reflection at least one of $(2,3),(2,4),(5,3),(5,4)$ is used. If we have $(2,3)$ or $(2,4)$ then we kill 3 A's, which forces all other A's to be used, so focusing on edges and corners progressively we get

$$(5,2) \implies (8,2),(5,5) \implies (8,5) \implies (8,8) \implies (5,8)$$

so $\{5,8\}\times\{2,5,8\}$ are filled out, as desired. If we have $(5,3)$ or $(5,4)$ then it's similar; we kill 3 A's which forces $(2,2)\implies(2,5)\implies(2,8)$ and also $(8,2)\implies(8,5)\implies(8,8)$, so $\{2,8\}\times\{2,5,8\}$ are filled out. By PIE the answer is $2\cdot 3^{\left(\frac{(9-2)-2}{2}\right)}=60$ minus the number of guys with all squares $(2,2) \pmod 3$, so $60-9=51$.

*One may also draw a 3×3 subgrid and note that if a 3×3 square contains part of a boundary line in its interior, say in the ith row, then if $i=1$ or $i=3$, the ith row of the 3×3 subgrid intersects at most 1 other 3×3 square. On the other hand, if $i=2$, we can show that the first or third column must have such a line instead. (Why?) $\square$

This problem was proposed by Ray Li.

36. Let $ABCD$ be a nondegenerate isosceles trapezoid with integer side lengths such that $BC \parallel AD$ and $AB=BC=CD$. Given that the distance between the incenters of triangles ABD and ACD is $8!$, determine the number of possible lengths of segment AD.

Answer. $\boxed{337}$.

Solution. Darn apparently you can bash this, and you don't need the observation that the incenters are on the diagonals, but i think the NT part is still nice, because the bound that $AB\cdot 3>AD$ makes the counting work out perfectly.

We get $8! = AD - \frac{AB+AD-BD}{2} - \frac{DC+DA-AC}{2} = BD-AB$. Let $x=AB=BC=CD$ and $y=AD$; then $BD=\sqrt{x^2+xy}$. The requirement $3x>y$ eventually reduces everything to $x \mid 8!^2 = 2^{14}3^4 5^2 7^2$ such that $x>8!$, so the answer is $\frac{15\cdot 5\cdot 3\cdot 3-1}{2}=337$. $\square$

This problem was proposed by Ray Li.

37. Let M be a positive integer. At a party with 120 people, 30 wear red hats, 40 wear blue hats, and 50 wear green hats. Before the party begins, M pairs of people are friends. (Friendship is mutual.) Suppose also that no two friends wear the same colored hat to the party.

During the party, X and Y can become friends if and only if the following two conditions hold:

a) There exists a person Z such that X and Y are both friends with Z. (The friendship(s) between Z, X and Z, Y could have been formed during the party.)
b) X and Y are not wearing the same colored hat.

Suppose the party lasts long enough so that all possible friendships are formed. Let M_1 be the largest value of M such that regardless of which M pairs of people are friends before the party, there will always be at least one pair of people X and Y with different colored hats who are not friends after the party. Let M_2 be the smallest value of M such that regardless of which M pairs of people are friends before the party, every pair of people X and Y with different colored hats are friends after the party. Find M_1+M_2.

Answer. $\boxed{4749}$.

Clarifications.

- The definition of M_2 should read, "Let M_2 be the *smallest* value of M such that...". An earlier version of the test read "largest value of M".

Solution. View as a tripartite graph with components of size 30, 40, 50. The condition just means that connected components are preserved and become complete at the end (i.e. with all edges filled in).

For $M = 30 + 40 + 50 - 1$ we would be able to find a spanning tree of the complete tripartite graph (where every two edges are connected/people are friends). Furthermore, any connected graph on 120 vertices has at least 119 edges (achieved by a tree), so $M_1 = 118$.

On the other hand, the largest connected component (in terms of edges) not containing all vertices has $(50-1) \cdot 40 + (50-1) \cdot 30 + 40 \cdot 30 = 4630$ edges, whence $M_2 = 4631$.

Thus $M_1 + M_2 = 4749$. □

This problem was proposed by Victor Wang.

38. Triangle ABC has sides $AB = 25$, $BC = 30$, and $CA = 20$. Let P, Q be the points on segments AB, AC, respectively, such that $AP = 5$ and $AQ = 4$. Suppose lines BQ and CP intersect at R and the circumcircles of $\triangle BPR$ and $\triangle CQR$ intersect at a second point $S \neq R$. If the length of segment SA can be expressed in the form $\frac{m}{\sqrt{n}}$ for positive integers m, n, where n is not divisible by the square of any prime, find $m + n$.

Answer. $\boxed{166}$.

Solution. Let $t = \frac{1}{5}$. Invert about A with power $R^2 = tAB \cdot AC = AQ \cdot AB = AP \cdot AC$ (i.e. radius R) and reflect about the angle bisector of $\angle BAC$ to get $C' = P$, $Q' = B$, $B' = Q$, $P' = C$. Since $(ABSQ)$ and $(ACSP)$ are cyclic, $S' = B'Q' \cap C'P' = QB \cap PC = R$. Thus $AS = \frac{R^2}{AS'} = \frac{R^2}{AR}$. By Ceva's theorem, $R \in AM$ where M is the midpoint of BC. Yet Menelaus yields

$$\frac{RA}{MA}\frac{MB}{CB}\frac{CP}{RP} = 1 \implies \frac{AR}{AM} = 2\frac{PQ}{PQ + BC} = \frac{2t}{t+1}.$$

But it's well-known that $4AM^2 + a^2 = 2(b^2 + c^2)$, so

$$SA = \frac{tbc}{\frac{2t}{1+t}AM} = (1+t)\frac{bc}{2AM} = \frac{6}{5}\frac{(5\cdot 5)(5\cdot 4)}{\sqrt{2(5\cdot 5)^2 + 2(5\cdot 4)^2 - (5\cdot 6)^2}} = 6\frac{(5)(4)}{\sqrt{2(5)^2 + 2(4)^2 - 6^2}} = \frac{120}{\sqrt{46}},$$

giving us an answer of $120 + 46 = 166$. □

Comment. Inspired by 2009 Balkan Math Olympiad Problem 2. We can also use complex numbers (based on spiral similarity).

This problem was proposed by Victor Wang.

39. Find the number of 8-digit base-6 positive integers $(a_1a_2a_3a_4a_5a_6a_7a_8)_6$ (with leading zeros permitted) such that $(a_1a_2\dots a_8)_6 \mid (a_{i+1}a_{i+2}\dots a_{i+8})_6$ for $i = 1, 2, \dots, 7$, where indices are taken modulo 8 (so $a_9 = a_1$, $a_{10} = a_2$, and so on).

Answer. $\boxed{40}$.

Solution. Call such an integer *good*; then $n = (a_1 a_2 \dots a_8)_6$ is good if and only if

$$n \mid \gcd(a_1, a_2, \dots, a_8)(6^8 - 1).$$

(Why?) Call a good number with gcd of digits equal to 1 *primitive.*

Now note that $6^8 - 1 = (6-1)(6+1)(6^2+1)(6^4+1)$ and $6-1$, $6+1$, 6^2+1, 6^4+1 are all prime, so every positive divisor d of $6^8 - 1$ has all ones and zeros if $6 - 1 = 5 \nmid d$ and all five and zeros if $6 - 1 = 5 \mid d$. (Prove this—think binary!) But $\frac{6^8-1}{6-1}$ has exactly $2^3 = 8$ positive divisors, so there are exactly 8 primitive good numbers. Furthermore, these 8 numbers have digits all equal to 0 or 1, so we can establish a 1-to-5 correspondence between the set of primitive good numbers and the set of good numbers (each primitive one corresponds to its first 5 multiples). Hence there are exactly $8 \cdot 5 = 40$ good numbers. $\square$

Comment. Inspired by `http://en.wikipedia.org/wiki/Cyclic_number`.

This problem was proposed by Victor Wang.

40. Let ABC be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Let M be the midpoint of BC and let Γ be the circle passing through A and tangent to line BC at M. Let Γ intersect lines AB and AC at points D and E, respectively, and let N be the midpoint of DE. Suppose line MN intersects lines AB and AC at points P and O, respectively. If the ratio $MN : NO : OP$ can be written in the form $a : b : c$ with a, b, c positive integers satisfying $\gcd(a, b, c) = 1$, find $a + b + c$.

Answer. $\boxed{225}$.

Solution. Menelaus on $\triangle MOC$ and $\triangle NOE$ give $\frac{MP}{OP}\frac{OA}{CA}\frac{CB}{MB} = 1$ and $\frac{NP}{OP}\frac{OA}{EA}\frac{ED}{ND} = 1$, respectively; dividing yields $\frac{MP}{NP} = \frac{CA}{EA}$, whence power of a point gives $\frac{MP}{MN} = \frac{CA}{CE} = \frac{CA^2}{CM^2}$. Similarly, $\frac{MO}{MN} = \frac{BA}{BD} = \frac{BA^2}{BM^2}$. It's now easy to compute $MN : NO : OP = a^2 : 4c^2 - a^2 : 4b^2 - 4c^2 = 49 : 120 : 56$. $\square$

This problem was proposed by James Tao.

41. While there do not exist pairwise distinct real numbers a, b, c satisfying $a^2+b^2+c^2 = ab+bc+ca$, there do exist complex numbers with that property. Let a, b, c be complex numbers such that $a^2 + b^2 + c^2 = ab + bc + ca$ and $|a + b + c| = 21$. Given that $|a - b| = 2\sqrt{3}$, $|a| = 3\sqrt{3}$, compute $|b|^2 + |c|^2$.

Clarifications.

- The problem should read $|a + b + c| = 21$. An earlier version of the test read $|a + b + c| = 7$; that value is incorrect.
- $|b|^2 + |c|^2$ should be a positive integer, not a fraction; an earlier version of the test read "…for relatively prime positive integers m and n. Find $m + n$."

Answer. $\boxed{132}$.

Solution. Equilateral triangle $\triangle abc$ exists if and only if the center $g = \frac{a+b+c}{3}$ exists. More precisely, the condition on (a, g) is that $|a| = 3\sqrt{3}$, $|g| = 7$, and $|g - a| = \frac{2\sqrt{3}}{2\cos 30^\circ} = 2$. By the triangle inequality, this can happen if and only if $|3\sqrt{3} - 7| \le 2 \le 3\sqrt{3} + 7$, which is indeed true.

Now that we've proven existence, note that a, b, c form the vertices of an equilateral triangle, so $|a - b| = |b - c| = |c - a|$, whence

$$3(|a|^2 + |b|^2 + |c|^2) = |a - b|^2 + |b - c|^2 + |c - a|^2 + |a + b + c|^2 = 3(2\sqrt{3})^2 + 21^2 = 477.$$

Thus $|b|^2 + |c|^2 = \frac{477}{3} - (3\sqrt{3})^2 = 132$. $\square$

Comment. A straightforward way to prove that a, b, c form the vertices of an equilateral triangle in the complex plane is to use the quadratic formula. (Try it!) Otherwise, it's rather difficult to discover the factorization $(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$ of $a^2 + b^2 + c^2 - ab - bc - ca$, where ω is a primitive third root of unity.

This problem was proposed by Ray Li.

42. Find the remainder when
$$\prod_{i=0}^{100}(1-i^2+i^4)$$
is divided by 101.

Answer. $\boxed{9}$.

Solution. Let $\omega = e^{2\pi i/12}$ and $z = e^{2\pi i/100}$; the key idea is that the symmetric sums of $(x-g)(x-g^2)\cdots(x-g^{100})$ and $(x-z)(x-z^2)\cdots(x-z^{100})$ are congruent mod 101, so the product is congruent mod 101 to the result (an integer) of
$$\prod_{j=1}^{100}(z^{4j}-z^{2j}+1) = \prod_{j=1}^{100}(z^j-\omega)(z^j-\omega^5)(z^j-\omega^7)(z^j-\omega^{11}).$$
Using the factorization of $x^{100}-1$, this boils down to
$$(\omega^{100}-1)(\omega^{500}-1)(\omega^{700}-1)(\omega^{1100}-1) = (\omega^4-1)^2(\omega^8-1)^2 = 3^2,$$
since ω^4, ω^8 are the two primitive 3rd roots of unity. $\square$

Solution 2. Primitive roots suffice. Let g be a primitive root modulo 101. Then, the product can be written in the form
$$\prod_{\substack{1\le i\le 100 \\ i^2\not\equiv -1 \pmod{101}}} \frac{i^6+1}{i^2+1}$$
Now because $g^{100}=1$, we can show that the multisets $\{i^6 \mid 1\le i\le 100\}$ and $\{i^2 \mid 1\le i\le 100\}$ are equivalent modulo 101. If we delete the elements -1 (which each appear twice) from both, the sets are still the same; in particular, their products are the same. Finally, this product is not zero. So, in our original product, almost all of the numerator and denominator cancel out except for the zeros, which we handle manually:
$$\frac{10^6+1}{10^2+1}\cdot\frac{90^6+1}{90^2+1} = (10^4-10^2+1)(90^4-90^2+1) \equiv 3\cdot 3 = 9 \pmod{101}. \qquad \square$$

This problem was proposed by Victor Wang. The second solution was given by Evan Chen.

43. In a tennis tournament, each competitor plays against every other competitor, and there are no draws. Call a group of four tennis players "ordered" if there is a clear winner and a clear loser (i.e., one person who beat the other three, and one person who lost to the other three.) Find the smallest integer n for which any tennis tournament with n people has a group of four tennis players that is ordered.

Answer. $\boxed{8}$.

Solution. First note that a group of four is "ordered" iff it's transitive iff it's acyclic (this follows more generally from the fact that if we contract all maximal directed cycles in a tournament, we get a transitive graph–Prove it!). The answer is 8. For $n=7$, we can take the tournament described in the last paragraph here (which also happens to be a counterexample for $k=2$ here—BTW, check out HroK's blog if you have not seen it before)—vertices are residues mod 7, and $x \to \{x+1, x+2, x+4\}$ give us the edges. For any vertex v, its out-neighbors $v+1, v+2, v+4$ form a triangle, which means there's no transitive set of four vertices. For $n=8$ there exists a vertex with at least $\frac{8-1}{2} = 3.5$

out-neighbors and thus some v has out-degree at least 4. But every 4-tournament has at least one transitive set of 3 vertices (else all 3-cycles), so we're done. $\square$

Comment. As Andre Arslan notes, the construction for $n = 7$ is unique up to isomorphism. You have to have two disjoint 3-cycles (take any 3-cycle out to leave four vertices, and there must be another 3-cycle among them). Because the sum of all out-degrees is 21 and nothing has out-degree 4, all out-degrees are 3. Now WLOG pick a vertex in a 3-cycle and two vertices in the other 3-cycle, so that the first of the three picked vertices beats the other two, but loses to the vertex in the other 3-cycle not picked. The edge relationships between the 3-cycles can now be completely determined (it's pretty easy to do out, there isn't really any casework) by looking at different groups of four in the group of six vertices alone. The directions of edges adjacent to the seventh vertex are then easily determined as well.

This problem was proposed by Ray Li.

44. Suppose tetrahedron $PABC$ has volume 420 and satisfies $AB = 13$, $BC = 14$, and $CA = 15$. The minimum possible surface area of $PABC$ can be written as $m+n\sqrt{k}$, where m, n, k are positive integers and k is not divisible by the square of any prime. Compute $m + n + k$.

Answer. $\boxed{346}$.

Solution. Heron's formula implies $[ABC] = \sqrt{21(21-13)(21-14)(21-15)} = 84$, whence the volume condition gives

$$420 = V(PABC) = \frac{h}{3}[ABC] = 28h \implies h = 15,$$

where h denotes the length of the P-altitude in tetrahedron $PABC$.

Let Q be the projection of P onto plane ABC and let $\triangle XYZ$ be the pedal triangle of Q with respect to $\triangle ABC$ (so X is the foot from Q to BC, etc.). Now define x, y, z to be the directed lengths QX, QY, QZ, respectively, so that $x > 0$ when Q and A are on the same side of BC and $x < 0$ otherwise (likewise for y, z). Observe that the only constraint on x, y, z is $ax+by+cz = 2[ABC] = 168$, because if reals x, y, z satisfy $ax+by+cz = 168$, there exists a point Q in plane ABC with $x = QX$, $y = QY$, $z = QZ$. As T varies along line BC, PT is minimized at $T = X$ (Why?), so $PX \perp BC$ and similarly, $PY \perp CA$ and $PZ \perp AB$. If we define the convex function $f(t) = \sqrt{t^2 + h^2} = \sqrt{t^2 + 15^2}$ for real t, then $PX = f(x)$, $PY = f(y)$, and $PZ = f(z)$. Thus the surface area of tetrahedron $PABC$ is just

$$[ABC] + \frac{af(x) + bf(y) + cf(z)}{2},$$

which by weighted Jensen is at least

$$[ABC] + \frac{1}{2}(a+b+c)f\left(\frac{ax+by+cz}{a+b+c}\right) = 84 + 21f(4) = 84 + 21\sqrt{241},$$

with equality at $x = y = z = \frac{168}{a+b+c} = 4$. $\square$

This problem was proposed by Ray Li.

45. Let N denote the number of ordered 2011-tuples of positive integers $(a_1, a_2, \ldots, a_{2011})$ with $1 \leq a_1, a_2, \ldots, a_{2011} \leq 2011^2$ such that there exists a polynomial f of degree 4019 satisfying the following three properties:

- $f(n)$ is an integer for every integer n;
- $2011^2 \mid f(i) - a_i$ for $i = 1, 2, \ldots, 2011$;
- $2011^2 \mid f(n + 2011) - f(n)$ for every integer n.

Find the remainder when N is divided by 1000.

Answer. $\boxed{211}$.

Solution. Throughout this proof, we will use finite differences and Newton interpolation. (See Lemma 1 in the first link for a complete proof.)

Note that $p = 2011$ is prime. We will show more generally that when we replace 4019 by $2p-3$ and 2011^2 by p^2, there are p^{2p-1} such p-tuples.

Lemma. If p is a prime, the congruence $(-1)^k p \equiv (-1)^{p-1}\binom{2p-2}{k+p} - \binom{2p-2}{k} \pmod{p^2}$ holds for $k = 0, 1, \dots, p-1$.

Proof. Expand. $\square$

Corollary. The $2p-2$th finite differences of the p-periodic sequence $b = (b_0, b_1, b_2, \dots)$ all vanish if and only if $p \mid b_1 + b_2 + \cdots + b_p$.

Proof. By the lemma,

$$\Delta^{2p-2}[b]_j = \sum_{i=0}^{2p-2} (-1)^i \binom{2p-2}{i} b_{j+i} \equiv -p \sum_{i=0}^{p-1} b_{j+i} \pmod{p^2}$$

for fixed $j \geq 0$. $\square$

Extend the indices of the a_i modulo p so that $a_i = a_{i \pmod{p}}$ for all integers i. Since $f(x)$ satisfies the three properties if and only if $f(x) + p^2 x^{2p-3}$ does, we just have to find the number of p-tuples interpolated by a polynomial of degree *at most* $2p-3$. By Newton interpolation, this is possible if and only if the $2p-2$th finite differences of the infinite (periodic) sequence $a = (a_0, a_1, a_2, \dots)$ all vanish modulo p^2. By the Lemma, this occurs when and only when $p \mid a_1 + a_2 + \cdots + a_p$, so the answer is

$$\frac{1}{p}(p^2)^p = p^{2p-1} = 2011^{4021} \equiv 11^{21} \equiv 1 + \binom{21}{1}10 + \binom{21}{2}10^2 \equiv 211 \pmod{1000}. \qquad \square$$

Solution 2. A more conceptual solution can be found by relating $(x-1)^{d+1}(a_0 + a_1 x + \cdots + a_{p-1}x^{p-1})$ modulo $p^2, x^p - 1$ to the "interpolating polynomials" of the p-tuple $(a_1, a_2, \dots, a_p)$. If $(x-1)^{2p-2} \equiv m(1 + x + \cdots + x^{p-1})$ in this modulus, it's not hard to prove $v_p(m) = 1$.

The idea is that 4020 is the smallest integer such that $(x-1)^{4020+1}$ vanishes mod $2011^2, x^{2011} - 1$, so you note that the interpolating polynomials of degree 4020 for $(1, 0, \dots, 0)$, $(0, 1, \dots, 0)$, etc., say $G_1, G_2, \dots, G_p$ (2011 places for the 1, all else 0) all have the same "Newton interpolation leading coefficient" (i.e. the coefficient of $\binom{x}{4020}$, or the 4020th finite difference), with 2011-adic valuation exactly 1. Indeed, for $(p, 0, \dots, 0)$, the least interpolating degree is lower than the maximum of 4020 (by induction, considering the mod $p, x^p - 1$ problem instead of mod $p^2, x^p - 1$), so this gives us $v_p = 1$, and of course, the 4020th finite differences of G_i all have to be equal since 4021th vanish always. So we have degree 4020 exactly when $2011 \nmid a_1 + \cdots + a_{2011}$, so the answer is $(1 - 1/p)(p^2)^p$. $\square$

Comment. See this thread and its resolution on MathOverflow for the original context of this problem.

The author conjectures that $\sum_{k=0}^{j} (-1)^{kp^i} \binom{p^i + (j-1)\phi(p^i) - 1}{kp^i} \equiv (-p)^{j-1} \pmod{p^j}$, which for $j = 2$ is stronger than the $v_p = 1$ condition mentioned in the second solution (it gives the exact value mod p^2). For higher j the reader can check this is stronger than the corresponding $v_p = j - 1$.

This problem was proposed by Victor Wang.

46. Let ABC be a triangle with $\angle B - \angle C = 30^\circ$. Let D be the point where the A-excircle touches line BC, O the circumcenter of triangle ABC, and X, Y the intersections of the altitude from A with the incircle with X in between A and Y. Suppose points A, O and D are collinear. If the ratio $\frac{AO}{AX}$ can be expressed in the form $\frac{a+b\sqrt{c}}{d}$ for positive integers a, b, c, d with $\gcd(a, b, d) = 1$ and c not divisible by the square of any prime, find $a + b + c + d$.

Answer. $\boxed{11}$.

Solution. The angle condition is equivalent to $\angle ADB = 60^\circ$. Let H be the orthocenter, I the incenter, and E the tangency point of BC and the incircle (I); reflect E over E' to get I. By the well-known homothety about A taking (I) to the A-excircle, we see that A, E', D are collinear. Thus $AE'OD$ is the A-isogonal of $AXHY$, so $AE' = AE$. Since $E'IE \perp BC \perp AXHY$ and $\angle DAH = 30^\circ$, $\angle AE'I = 150^\circ$ and since $IX = IE'$, $\triangle AXI \cong \triangle AE'I$, whence $\angle AXI = 150^\circ$ and thus $AXIE'$ is a rhombus. Now note that $OE = OD$, so O is the midpoint of $E'D$ in right triangle $\triangle E'ED$. But I is the midpoint of $E'E$, so $\angle OIE' = 90^\circ$ and it follows that

$$\frac{AO}{AX} = \frac{AO}{AE'} = 1 + \frac{OE'}{IE'} = 1 + \frac{1}{\sin 60^\circ} = \frac{3+2\sqrt{3}}{3}. \qquad \square$$

Solution 2. Let AD meet the incircle ω at X', and recall that $X'I$ is a diameter of ω. Let the inradius be r and the circumradius R. Note that since $\angle BAY = \angle DAC = 90 - B$, we can derive that $\angle XAX' = 30^\circ$ and $AX = AX'$.

Then, note also that O is the midpoint of $X'D$. Using the angle information, we find that $IO = \frac{r}{\sqrt{3}}$. Observe $OA = OB = R$.

Assume WLOG that $r = \sqrt{3}$. Now, set $I = (-1, 0)$, $O = (0, 0)$, $A = \left(-\frac{R}{2}, \frac{\sqrt{3}R}{2}\right)$ and $B = \left(-\sqrt{R^2-3}, -\sqrt{3}\right)$. We need AB to be tangent to ω; that is, the distance from I to AB is $r = \sqrt{3}$. A few pages of calculation yield the quartic $R^4 - 11R^2 - 12R + 4 = 0$, which has a double root at -2 and gives $(R+2)^2(R^2 - 4R + 1) = 0$, which gives $R = \sqrt{3} + 2$ as the only valid solution.

Then, $\frac{AO}{AX'} = \frac{R}{R - \frac{2}{\sqrt{3}}r} = \frac{2+\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}+3}{3}$. $\square$

Solution 3. (Sketch) It's possible to find a trigonometric solution with only the synthetic observation $AE = AE'$. Indeed, we can use the homothety relation $\frac{AE'}{AD} = \frac{r}{r_a} = \frac{s-a}{s}$ and the law of sines on $\triangle ADC$ to get everything in terms of trigonometric functions involving C. Using the sine and cosine addition and subtraction formulas a few times, we can get an equation in terms of $\cos 2x$ and $\cos x$, where $x = C + 15^\circ = 90^\circ - \frac{A}{2}$. $\square$

This problem was proposed by James Tao. The second solution was provided by Evan Chen. The third solution was provided by Victor Wang.

47. Let $f(x, y)$ be a function from ordered pairs of positive integers to real numbers such that

$$f(1, x) = f(x, 1) = \frac{1}{x} \quad \text{and} \quad f(x+1, y+1)f(x, y) - f(x, y+1)f(x+1, y) = 1$$

for all ordered pairs of positive integers (x, y). If $f(100, 100) = \frac{m}{n}$ for two relatively prime positive integers m, n, compute $m + n$.

Answer. $\boxed{111088900001}$.

Solution. We can easily prove by induction that

$$f(x,y) = \frac{1+4\binom{x+1}{3}\binom{y+1}{3}}{xy}$$

for all positive integers x, y. It quickly follows that $f(100,100) = \frac{1+111100\cdot 999900}{10000}$. $\square$

Comment. Perhaps the most natural way to find the pattern is to find f for small fixed values of y (say 0 through 3). When searching for the pattern, it is helpful to substitute $g(x,y) = xyf(x,y)$.

This problem was proposed by David Yang.

48. ω is a complex number such that $\omega^{2013} = 1$ and $\omega^m \neq 1$ for $m = 1, 2, \dots, 2012$. Find the number of ordered pairs of integers (a, b) with $1 \le a, b \le 2013$ such that

$$\frac{(1+\omega+\cdots+\omega^a)(1+\omega+\cdots+\omega^b)}{3}$$

is the root of some polynomial with integer coefficients and leading coefficient 1. (Such complex numbers are called *algebraic integers*.)

Answer. $\boxed{4029}$.

Solution. We will repeatedly use the fact that the set AI of algebraic integers is closed under addition and multiplication (in other words, AI is a *ring*). First, however, we show that

$$\frac{(1+\omega+\cdots+\omega^a)(1+\omega+\cdots+\omega^b)}{3}$$

must in fact be an algebraic integer for *every* primitive 2013th root (if it is for at least one)[1]. Actually, we can prove a much more general statement.

Lemma 1. Let z be an algebraic number with minimal polynomial $f \in \mathbb{Q}[x]$ of degree $n \ge 1$ (so that f is monic). Suppose f has roots $z_1 = z, z_2, \dots, z_n$ (these are the *conjugates* of z). If there exists a polynomial $g \in \mathbb{Q}[x]$ for which $g(z) \in AI$, then $g(z_i) \in AI$ for $i = 1, 2, \dots, n$.

Proof. Let $h \in \mathbb{Q}[x]$ be the minimal polynomial of $g(z)$; then h (which is monic) has integer coefficients since $g(z) \in AI$. Since $h(g(x))$ has rational coefficients and $h(g(z)) = 0$, the minimal polynomial $f(x)$ of z must divide $h(g(x))$. But then $h(g(z_i)) = 0$ for all i (the z_i are the roots of f by definition), so $g(z_i) \in AI$ for all i (the roots of $h(x)$ are all algebraic integers by definition).

For a slightly different (but really equivalent) perspective, consider the map $f_i : \mathbb{Q}(z) \to \mathbb{Q}(z_i)$ that fixes the rationals, is multiplicative and additive, and maps z to z_i; then f_i is a ring isomorphism (because z and z_i have the same minimal polynomial). In particular, since f_i fixes the coefficients of g and h, we have

$$0 = f_i(0) = f_i(h(g(z))) = h(g(f_i(z))) = h(g(z_i)) = 0,$$

whence z_i is also a root of $h(g(x))$.

See the second post here for a more "concrete" application of these ideas. $\square$

Lemma 2. The nth cyclotomic polynomial $\Phi_n(x)$ evaluated at 1 gives 0 for $n = 1$, p for $n = p^k$ a prime power, and 1 otherwise.

[1] For contest purposes, this can actually just be deduced by meta-analysis of the problem statement.

Proof. From $\prod_{d|n} \Phi_d(x) = x^n - 1$, we get $\prod_{1<d|n} \Phi_d(x) = 1 + x + \cdots + x^{n-1}$. The rest follows by induction on n. (While this is not strictly necessary, observe that for $n > 1$, $\Phi_n(1) \neq 0$ since 1 is not a primitive nth root of unity.) $\square$

Corollary. If n has at least two distinct prime factors, $1 - \zeta$ is an "invertible" algebraic integer for any primitive nth root of unity ζ.

Proof. First suppose n has at least two distinct prime factors. Clearly $1 - \zeta$ is an algebraic integer. On the other hand, Lemma 2 gives us $\Phi_n(x) = (x-1)f(x) + 1$ for some polynomial $f \in \mathbb{Z}[x]$, so plugging in $x = \zeta$ gives $(1-\zeta)^{-1} = f(\zeta)$, which is indeed an algebraic integer (as f has integer coefficients). $\square$

The corollary tells us that for a fixed 2013th primitive root of unity ω,

$$\frac{(1 + \omega + \cdots + \omega^a)(1 + \omega + \cdots + \omega^b)}{3} \in AI$$

if and only if $\frac{(1-\omega^{a+1})(1-\omega^{b+1})}{3} \in AI$ is too. From now on we work with this simpler form.

If $2013 \mid a+1$ or $2013 \mid b+1$, the expression vanishes and is trivially an algebraic integer, so now suppose $2013 \nmid a+1$ and $2013 \nmid b+1$. Multiplying the new expression over all 2013th primitive roots of unity shows that

$$3^{-\phi(2013)} \prod_{\substack{0<k<2013 \\ \gcd(k,2013)=1}} (1 - e^{2\pi k(a+1)i/2013})(1 - e^{2\pi k(b+1)i/2013})$$

must be both a rational number (Why?) and an algebraic integer, and thus a "rational" integer by the rational root theorem (*). On the other hand,

$$\prod_{\substack{0<k<2013 \\ \gcd(k,2013)=1}} (1 - e^{2\pi k(a+1)i/2013}) = (\Phi_c(1))^{\phi(2013)/\phi(c)},$$

where $c = 2013/\gcd(2013, a+1) > 1$ by our assumption $2013 \nmid a+1$ (similarly define $d > 1$ in terms of b). However, since 2013 is square-free, and $\phi(c), \phi(d) \geq 2$ (since $c, d > 1$ and $c, d \mid 2013$), Lemma 2 forces $c = d = 3$, or else we contradict (*) (as we fail to cancel out the factors of 3 earlier).

To prove that the expression is an algebraic integer whenever $c = d = 3$, we note the identities $\frac{(1-\zeta)^2}{3} = -\zeta$ and $\frac{(1-\zeta)(1-\zeta^2)}{3} = 1$ (which are both algebraic integers) for primitive 3rd roots of unity ζ. More generally, this follows from the fact that $\frac{1-\zeta}{\sqrt{3}}$ is an algebraic integer (in fact, an invertible algebraic integer).

Thus (a, b) works if and only if $2013 \mid a+1$, $2013 \mid b+1$, or $(a+1, b+1) \in \{2013/3, 2 \cdot 2013/3\}^2$, so the desired total is $(2013^2 - 2012^2) + 2^2 = 4029$. $\square$

Comment. The interested reader may want to further investigate the "Corollary" stated above for $n = p^k$ a prime power. For instance, $(1-\zeta)^{-1}$ is not an algebraic integer, but $p(1-\zeta)^{-1}$ is. (Why?) How about $p^{-1}(1-\zeta)$? (It may help to think about the $p = 3$ case introduced in this problem.)

This problem was proposed by Victor Wang.

49. In $\triangle ABC$, $CA = 1960\sqrt{2}$, $CB = 6720$, and $\angle C = 45°$. Let K, L, M lie on BC, CA, and AB such that $AK \perp BC$, $BL \perp CA$, and $AM = BM$. Let N, O, P lie on KL, BA, and BL such that $AN = KN$, $BO = CO$, and A lies on line NP. If H is the orthocenter of $\triangle MOP$, compute HK^2.

Clarifications.

- Without further qualification, "XY" denotes line XY.

Answer. $\boxed{68737600}$.

Solution. Let $a = BC$, $b = CA$, $c = AB$, and compute $c = 3640\sqrt{2}$. Let M' be the midpoint of AC and let O' be the circumcenter of $\triangle ABC$.

It is well known that $KMLM'$ is cyclic, as is $AMO'M'$. Also, $\angle BO'A = 2\angle C = 90^\circ$, so O' lies on the circle with diameter AB. Then N is the radical center of these three circles; hence A, N, O' are collinear.

Now applying Brokard's Theorem to the circle with diameter AB, we find that M is the orthocenter of the OPH', where $H' = LA \cap BO'$. Hence H' is the orthocenter of $\triangle MOP$, whence $H = H' = AC \cap BO'$.

Then, it is well known[2] that

$$\frac{AH}{HC} = \frac{c^2(a^2+b^2-c^2)}{a^2(b^2+c^2-a^2)}$$

where the lengths are directed. Cancelling a factor of 280^2 we can compute:

$$\begin{aligned}\frac{AH}{HC} &= \frac{c^2(a^2+b^2-c^2)}{a^2(b^2+c^2-a^2)}\\ &= \frac{338(576+98-338)}{576(98+338-576)}\\ &= -\frac{169}{120}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{AC}{HC} &= 1 + \frac{AH}{HC}\\ &= -\frac{49}{120}\\ \implies |HC| &= \frac{120}{49} \cdot 1960\sqrt{2}\\ &= 4800\sqrt{2}.\end{aligned}$$

Now applying the Law of Cosines to $\triangle KCH$ with $\angle KCH = 135^\circ$ yields

$$\begin{aligned}HK^2 &= KC^2 + CH^2 - 2KC \cdot CH \cdot \cos 135^\circ\\ &= 1960^2 + \left(4800\sqrt{2}\right)^2 - 2(1960)\left(4800\sqrt{2}\right)\left(-\frac{1}{\sqrt{2}}\right)\\ &= 40^2\left(49^2 + 2 \cdot 120^2 + 2 \cdot 49 \cdot 120\right)\\ &= 1600 \cdot 42961\\ &= \boxed{68737600}.\end{aligned}$$

$\square$

Solution 2. We use the following more general formulation:

[2] One can prove this just using the identity $\frac{BP}{PC} = \frac{b \sin \angle PAB}{c \sin \angle CAP}$ and substituting the law of cosines in.

> Work in the projective plane for convenience and consider any triangle ABC. Let K, L be the feet of the altitudes from A, B respectively, and let $T = AK \cap BL$ be the orthocenter of $\triangle ABC$. Let P be the intersection of line BLT and the tangent AA to (ATB) (this line contains N in the original problem, so the definition of P is still the same regardless of angle C), and let O be the point on line AB satisfying $LO \perp BC$ (the new definition of O is the key change). If M is the midpoint of AB, compute HK^2.

Let $Q = LO \cap BC$ be the foot from L to BC. We want to relate P to O, M somehow; define S to be the intersection of lines $PA(N)$ and LOQ. Since lines $LOQS, TAK$ are parallel and PA is tangent to (ATB),

$$\measuredangle LSA = \measuredangle TAP = \measuredangle TBA = \measuredangle LBA,$$

so L, A, K, S, B all lie on the circle ω with diameter AB. By Brokard's theorem, $P = AS \cap BL$, $O = AB \cap LS$, $X = AL \cap BS$ form a self-polar triangle, so M, the center of ω, is the orthocenter of $\triangle XOP$. But then X must be the orthocenter of $\triangle MOP$, so in fact $X = H$. Now proceed as above. For a directed-friendly finish, use the law of sines on $\triangle HBC$ to get HB and HC in terms of BC and $\angle BHC$, and then use the directed version of Stewart's theorem to get HK. $\square$

This problem was proposed by Evan Chen. The second solution was provided by Victor Wang.

50. Let S denote the set of words $W = w_1 w_2 \ldots w_n$ of any length $n \geq 0$ (including the empty string λ), with each letter w_i from the set $\{x, y, z\}$. Call two words U, V *similar* if we can insert a string $s \in \{xyz, yzx, zxy\}$ of three consecutive letters somewhere in U (possibly at one of the ends) to obtain V or somewhere in V (again, possibly at one of the ends) to obtain U, and say a word W is *trivial* if for some nonnegative integer m, there exists a sequence $W_0, W_1, \ldots, W_m$ such that $W_0 = \lambda$ is the empty string, $W_m = W$, and W_i, W_{i+1} are similar for $i = 0, 1, \ldots, m-1$. Given that for two relatively prime positive integers p, q we have

$$\frac{p}{q} = \sum_{n \geq 0} f(n) \left(\frac{225}{8192}\right)^n,$$

where $f(n)$ denotes the number of trivial words in S of length $3n$ (in particular, $f(0) = 1$), find $p + q$.

Answer. $\boxed{61}$.

Solution. First we make some helpful definitions.

For convenience, call xyz, yzx, zxy the three *cyclic words*. Let λ be the empty string. Say the word W is *reduced* unless and only unless at least one of the three cyclic words appears as a string of three consecutive letters in W. (In particular, λ is reduced.) Let $S^* \subseteq S$ be the set of reduced words.

Define $\sim$ so that $U \sim V$ if and only if U, V are similar. Call two words U, V *equivalent* if there exists a finite sequence of words $W_0, W_1, \ldots, W_m$ such that $U = W_0$, $V = W_m$, and $W_i \sim W_{i+1}$ for $i = 0, 1, \ldots, m-1$. Define the binary relation $\equiv$ so that $U \equiv V$ if and only if U, V are equivalent. Then $\equiv$ is clearly reflexive, symmetric, and transitive, and W is trivial if and only if $W \equiv \lambda$. Furthermore, if $A \equiv C$ and $B \equiv D$, then $AB \equiv CD$.

The distinction between $=$ and $\equiv$ will be important throughout the proof. We have $U = V$ if and only if U, V are identical when parsed as *sequences of letters*, i.e. they're composed of the same letters in the same order (and in particular, have the same length).

Let T be the set of trivial words and $T_0 \subseteq T$ be the set of *minimal* trivial words, i.e. words $W = w_1 w_2 \ldots w_n \in T$ such that $w_1 w_2 \ldots w_i \notin T$ for $i = 1, 2, \ldots, n-1$ (we vacuously have $\lambda \in T_0$). It's easy to see that every trivial word W can be uniquely expressed in the form $W_1 W_2 \ldots W_m$ for some nonnegative integer m and nonempty $W_1, W_2, \ldots, W_m \in T_0$. (*) Indeed, this follows from the simple fact that $B \equiv \lambda$ if $AB, A \in T$.

Lemma 1. No two distinct reduced words are equivalent.

Proof. Define a "canonical" reduced form $R(W)$ for each word $W \in S$ by always deleting the leftmost cyclic word in W until it's no longer possible (so $zxyxyzz$ becomes $xyzz$ and then z).

We will now show that $R(A) = R(B)$ whenever $A \sim B$. WLOG suppose that $A = PQ$ and $B = PxyzQ$ for two words P, Q (possibly empty). Since $R(A) = R(R(P)Q)$ and $R(B) = R(R(P)xyzQ)$, we can WLOG suppose that $P \in S^*$. If xyz is the leftmost cyclic word in B, then $R(PxyzQ) = R(PQ)$. Otherwise, P ends in z. If $P = P'yz$ for some $P' \in S$ (since $P \in S^*$, P' doesn't end in x), then yzx is the leftmost cyclic word in B and $R(PxyzQ) = R(P'(yzx)yzQ) = R(P'yzQ) = R(PQ)$. Finally, if $P = P'z$ for some $P' \in S$ but P' doesn't end in y, then zxy is the leftmost cyclic word in B and $R(PxyzQ) = R(P'(zxy)zQ) = R(P'zQ) = R(PQ)$, completing the proof that $R(A) = R(B)$.

We can directly extend this to find that $R(A) = R(B)$ whenever $A \equiv B$. But then $R(A) \neq R(B)$ implies $A \not\equiv B$ for any two distinct reduced words A, B. □

□

Comment. This is loosely modeled off of the proof in Lemma 1 here that the free group is nontrivial.

Corollary. $R(W)$ is the unique reduced word $W^* \in S^*$ equivalent to W. In particular, W is trivial if and only if $R(W) = \lambda$ (so $3 \mid |W|$ for all $W \in T$).

Proof. If not, then there are two distinct reduced words equivalent to W, contradicting the lemma. □

We are now prepared to prove the key lemma. Define the function $t : \{x, y, z\} \to \{x, y, z\}$ so that $t(x) = y$, $t(y) = z$, $t(z) = x$.

Lemma 2. Let n be a positive integer. Then the word $W = w_1 w_2 \dots w_n$ lies in T_0 if and only if there exists a sequence of nonempty words $A_1, A_2, \dots, A_m \in T_0$ ($m \geq 0$) and index $p \in \{0, 1, \dots, m\}$ such that

$$W = w_1 A_1 A_2 \dots A_p t(w_1) A_{p+1} A_{p+2} \dots A_m t^2(w_1),$$

none of $A_1, A_2, \dots, A_p$ end with w_1, and none of $A_{p+1}, A_{p+2}, \dots, A_m$ start with $t^2(w_1)$. Furthermore, this representation is unique for any fixed $W \in T_0$.

Proof. Call the representation a *standard presentation of* W if it exists.

We proceed by strong induction on $n \geq 1$. For the base case, note that by the Corollary (to Lemma 1)[3], the only nonempty trivial words of length at most three are the three cyclic words, which are also minimal. This proves the claim for $n \leq 3$.

Now suppose $n \geq 4$ and assume the inductive hypothesis for $1, 2, \dots, n-1$. We will first show that every minimal trivial word W has a standard presentation. Of course, the Corollary implies $3 \mid n$. By the minimality of W and the Corollary (consider the possible sequences of reductions taking W to $R(W) = \lambda$), there exists an index $q \in \{1, 2, \dots, n\}$ such that

$$w_1 w_q w_n \equiv w_2 w_3 \dots w_{q-1} \equiv w_{q+1} w_{q+2} \dots w_{n-1} \equiv \lambda.$$

(Why?) Then $w_q = t(w_1)$, $w_n = t^2(w_1)$, and by (*), there exists a (unique) sequence of nonempty words $A_1, A_2, \dots, A_m \in T_0$ ($m \geq 0$) and index $p \in [0, m]$ such that $w_2 \dots w_{q-1} = A_1 \dots A_p$ and $w_{q+1} \dots w_{n-1} = A_{p+1} \dots A_m$. If for some $i \in [p+1, m]$, A_i ends with $t^2(w_1)$, then

$$w_1 A_1 \dots A_p t(w_1) A_{p+1} \dots A_{i-1} t^2(w_1) \equiv \lambda,$$

[3] Lemma 1, while only explicitly used once, is the cornerstone of the base case, without which the induction would crumble. In particular, it shows that $xzy, yxz, zyx \notin T$.

contradicting the minimality of W. Similarly, if A_i ends with w_1 for some $i \in [1, p]$, then

$$w_1 A_{i+1} \dots A_p t(w_1) A_{p+1} \dots A_m t^2(w_1) \equiv \lambda,$$

once again contradicting the minimality of W (this time, however, we use the fact that W is minimal if and only if $w_i \dots w_n \not\equiv \lambda$ for $i \in [1, n]$, which follows, for instance, from (*)).

The previous paragraph establishes the existence of a standard presentation for $W \in T_0$. Now assume for contradiction that there exists another such representation

$$w_1 A_1' A_2' \dots A_{p'}' t(w_1) A_{p'+1}' A_{p'+2}' \dots A_{m'}' t^2(w_1),$$

with the $t(w_1)$ in the middle indexed at $w_{q'}$ (the $t^2(w_1)$ at the end is, of course, w_n). By (*), q' cannot equal q, so WLOG $q' > q$. We must then have $A_i' = A_i$ for $i = 1, 2, \dots, p$, whence A_{p+1}' starts with $t(w_1)$ and by the inductive hypothesis, ends with $t^2(t(w_1)) = w_1$. Yet this contradicts the validity of A_{p+1}', so the standard presentation of W must in fact be unique.

Finally, we show that if W has a standard presentation, then it must lie in T_0. Since $w_1 t(w_1) t^2(w_1) \equiv A_i \equiv \lambda$ for $i = 1, 2, \dots, m$, W is obviously trivial; it remains to show that W is minimal. Go by contradiction and suppose there exists $n' \in [1, n)$ such that $W' = w_1 w_2 \dots w_{n'} \in T_0$; by the inductive hypothesis, W' has a (unique) standard presentation

$$w_1 A_1' A_2' \dots A_{p'}' t(w_1) A_{p'+1}' A_{p'+2}' \dots A_{m'}' t^2(w_1).$$

As in the previous paragraph, if q' denotes the index of $t(w_1)$ in W, then we must have $q' = q$ (otherwise one of $A_1', \dots, A_{p'}', A_1, \dots, A_p$ starts with $t(w_1)$ and so ends with w_1). Therefore $m' < m$ and $A_i' = A_i$ for $i \in [p' + 1 = p + 1, m']$, whence $t^2(w_1)$ will appear at the start of $A_{m'+1}$, contradiction. $\square$

In view of Lemma 2, let $g(n)$ be the number of minimal trivial words of length $3n$ ($n > 0$). Then $g(1) = 3$ and considering standard presentations, we get

$$\frac{g(n)}{3} = \sum_{\substack{a_1 + \dots + a_m = n-1 \\ m, a_1, \dots, a_m \geq 1}} (m+1) \prod_{i=1}^{m} \frac{2}{3} g(a_i)$$

for[4] $n \geq 2$ (in the notation of Lemma 2, there are $m + 1$ choices for p, and $\frac{2}{3} g(|A_i|)$ choices for each A_i, regardless of whether $i \leq p$ or $i \geq p + 1$). Summing over $n \geq 2$ yields

$$\begin{aligned} \frac{G(u)}{3} &= \frac{g(1)u}{3} + u \sum_{m \geq 1} (m+1) \left(\frac{2}{3} G(u) \right)^m \\ &= u \sum_{m \geq 0} (m+1) \left(\frac{2}{3} G(u) \right)^m \\ &= \frac{u}{(1 - 2G(u)/3)^2}, \end{aligned}$$

where $G(u) = \sum_{n \geq 1} g(n) u^n$. By (*), $F(u) = \sum_{n \geq 0} f(n) u^n = \frac{1}{1 - G(u)}$.

Before evaluating the generating functions F, G at $u = 225/8192$, observe that $0 \leq g(n) \leq f(n) \leq 3^{3n}$ for every nonnegative integer n, so

$$0 < G(225/8192) \leq F(225/8192) \leq \sum_{n \geq 0} \left(3^3 \frac{225}{8192} \right)^n$$

[4]Actually, the recurrence is true for $n = 1$ "by convention," i.e. if we allow $m = 0$ and interpret empty products as 1. This simplifies the calculation of $G(u)$ slightly.

converges (as $3^3 \cdot 225 = 6075 < 8192$). Thus $G(u), F(u)$ are positive reals satisfying $G(u) = 1 - F(u)^{-1} < 1$.

Finally, letting $v = G(u)/3$, we get

$$\frac{15^2}{2^{13}} = \frac{225}{8192} = v(1-2v)^2 \implies 15^2 = w(16-w)^2,$$

where $w = 2^5 v$. Solving this cubic yields $w \in \{1, \frac{1}{2}(31 \pm \sqrt{61})\}$. But $\frac{3}{32}w = G(u) < 1 \implies w < \frac{32}{3} < 11 < \frac{1}{2}(31 \pm \sqrt{61})$, forcing $w = 1 \implies G(u) = \frac{3}{32}$. Therefore $F(u) = \frac{1}{1-G(u)} = \frac{1}{1-3/32} = \frac{32}{29}$.

Solution 2. Map every word by adding the position of the word to itself mod 3, so we can consider inserting/deleting xxx, yyy, or zzz instead.

Now consider each word as a sequence of instructions acting on a stack, where x is an instruction that says "if the top two elements of the stack are both x, pop both of them; otherwise push x". We claim the set of trivial words is the set of programs that, when run on an empty stack, result in another empty stack.

The three sequences xxx, yyy, zzz are always no-ops (they just cycle the number of some element directly on top of the stack mod 3), so clearly inserting or deleting them will preserve the effects of a program. Furthermore, any effectless program can be obtained by insertions alone: just look at the 2 operations before and 1 after any moment when the stack has the most elements, and induct on length.

Then we can note three important parts in each trivial nonempty program P: (I) after the first symbol, say w is pushed; (II) after the second copy of that symbol w is pushed; (III) after ww is popped.

The part after (III) can be any other trivial program. The parts between (I) and (II), and between (II) and (III), must each be trivial programs where w is never pushed as the bottom of the stack. Let's call these "w-bottomless" programs (all the good names have been taken :( ).

w-bottomless programs can be divided into three parts in the same way. Then we discover the necessary parts in each part are also something-bottomless programs. That is, you can get an x-bottomless program, say, by picking which of y or z to push (WLOG y), which y-bottomless program to put in part (I), which y-bottomless program to put in part (II), and which x-bottomless program to put in part (III). So there's a simple generating function equation:

$$B(x) = 1 + 2xB(x)^3.$$

You can actually count the number of w-bottomless programs directly. Factor out the power of 2 that results from each choice of symbol, so the difference sequence of the stack heights/sizes is a sequence of $+1$s and -2s that sum to 0 with no negative partial sums. Then insert a $+1$ at the start to convert the condition to "no *nonnegative* partial sums", cyclically permute the steps, and observe that exactly one cyclic permutation of these steps has no nonnegative partial sums, so we get a total of $2^n \frac{1}{3n+1}\binom{3n+1}{n} = \frac{2^n}{n}\binom{3n}{n-1}$ where n is the number of -2s, or equivalently, $3n$ is the number of steps in the w-bottomless program. To recover the stack sequence/bottomless program from the sequence of $+1$s and -2s, consider the first -2, which must have two $+1$s before it, and remove these three numbers. These three correspond to a "pop", and because they have zero sum, we can repeatedly take out $+1, +1, -2$ subsequences, each corresponding to a "layer" in the program. Order the layers as follow: the layer corresponding to the first $+1$ is the "first"; after that, the first $+1$ we encounter that is not in the first layer and its two partners belong to the "second layer", and so on. For convenience, define a "phantom zeroth layer" www. Then when we evaluate the program (including pops, etc.), the first $+1$ of the ith layer, $i > 0$ will be directly "above" a $+1$ (either the first or second) of the jth layer for some $0 \le j < i$, and thus must have a different symbol. (For $j = 0$, it just means we've reached the bottom again, so the ith layer can't have symbol w by the w-bottomless restriction). In particular, the first layer cannot start with w.) Thus we have 2^n ways to assign symbols to the sequence of $+1$s and -2s, one power of 2 for each layer.

Then for the original sequence's generating function, again: there are 3 ways to choose which symbol is first pushed, and you have to choose two something-bottomless programs for (I) and (II) and one trivial program for (III), so:

$$F(x) = 1 + 3xB(x)^2F(x)$$

Then just plug in x and solve the two equations for $B(x)$ and $F(x)$. (Presumably there are extraneous solutions, but I just assumed the one that looked nice was right. :P) $\square$

Comment. The key here is to recognize that this is sort of similar to Catalan parentheses recurrence (with the notion of minimal words). In fact if we replace $\{xyz, yzx, zxy\}$ with $\{xy\}$, then the problem is equivalent to Catalan. However, the primary motivation for this problem is the one posted here, with $x + y + y^{-1}x^{-1}$ instead of $x + y + x^{-1} + y^{-1}$. Also, it is possible to get a nice closed form for $g(n)$ using the Lagrange inversion theorem: we can get

$$g(n) = \frac{3 \cdot 2^{n-1}}{3n-1}\binom{3n-1}{n},$$

which checks out with the computations above using WolframAlpha.

However, in order to generalize this to four or more letters in the natural way (so $\{xyzw, yzwx, \ldots\}$ for four letters), we need to slightly modify Lemma 2. The key ideas are the same, however, so we leave the rest to the interested reader. (It may be helpful to use the fact that if $UV \equiv \lambda$, then $VU \equiv \lambda$. Basic group theory, particularly the notion of inverses, may make this slightly easier to see.)

This problem was proposed by Victor Wang. The second solution was given by Brian Chen, with a close variant posted by Team SA on AoPS.