The Online Math Open Winter Contest
January 4, 2013–January 14, 2013

Online Math Open

Acknowledgments

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Contest Information

Format

The test will start Friday, January 4 and end Monday, January 14. You will have until 7pm EST on January 14 to submit your answers. The test consists of 50 short answer questions, each of which has a nonnegative integer answer. The problem difficulties range from those of AMC problems to those of Olympiad problems. Problems are ordered in roughly increasing order of difficulty.

Team Guidelines

Students may compete in teams of up to four people. Participating students must not have graduated from high school. International students may participate. No student can be a part of more than one team. The members of each team do not get individual accounts; they will all share the team account.

Each team will submit its final answers through its team account. Though teams can save drafts for their answers, the current interface does not allow for much flexibility in communication between team members. We recommend using Google Docs and Spreadsheets to discuss problems and compare answers, especially if teammates cannot communicate in person. Teams may spend as much time as they like on the test before the deadline.

Aids

Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids are not allowed. This is includes (but is not limited to) Geogebra and graphing calculators. **Published print and electronic resources are not permitted.** (This is a change from last year's rules.)

Four-function calculators are permitted on the Online Math Open. That is, calculators which perform only the four basic arithmetic operations (+-*/) may be used. Any other computational aids such as scientific and graphing calculators, computer programs and applications such as Mathematica, and online databases are prohibited. All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.

Clarifications

Clarifications will be posted as they are answered. For the Fall 2012-2013 Contest, they will be posted at here. If you have a question about a problem, please email OnlineMathOpenTeam@gmail.com with "Clarification" in the subject. We have the right to deny clarification requests that we feel we cannot answer.

Scoring

Each problem will be worth one point. Ties will be broken based on the "hardest" problem that a team answered correctly. Remaining ties will be broken by the second hardest problem solved, and so on. Problem X is defined to be "harder" than Problem Y if and only if

(i) X was solved by less teams than Y, OR

(ii) X and Y were solved by the same number of teams and X appeared later in the test than Y.

Note: This is a change from prior tiebreaking systems. However, we will still order the problems by approximate difficulty.

Results

After the contest is over, we will release the answers to the problems within the next day. If you have a protest about an answer, you may send an email to OnlineMathOpenTeam@gmail.com (Include "Protest" in the subject). Solutions and results will be released in the following weeks.

1. Let x be the answer to this problem. For what real number a is the answer to this problem also $a-x$?

2. The number 123454321 is written on a blackboard. Evan walks by and erases some (but not all) of the digits, and notices that the resulting number (when spaces are removed) is divisible by 9. What is the fewest number of digits he could have erased?

3. Three lines m, n, and ℓ lie in a plane such that no two are parallel. Lines m and n meet at an acute angle of $14°$, and lines m and ℓ meet at an acute angle of $20°$. Find, in degrees, the sum of all possible acute angles formed by lines n and ℓ.

4. For how many ordered pairs of positive integers (a, b) with $a, b < 1000$ is it true that a times b is equal to b^2 divided by a? For example, 3 times 9 is equal to 9^2 divided by 3.

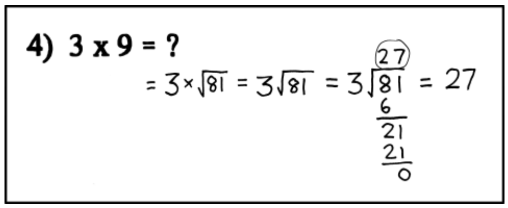

Figure 1: xkcd 759

5. At the Mountain School, Micchell is assigned a *submissiveness rating* of 3.0 or 4.0 for each class he takes. His *college potential* is then defined as the average of his submissiveness ratings over all classes taken. After taking 40 classes, Micchell has a college potential of 3.975. Unfortunately, he needs a college potential of at least 3.995 to get into the South Harmon Institute of Technology. Otherwise, he becomes a rock. Assuming he receives a submissiveness rating of 4.0 in every class he takes from now on, how many more classes does he need to take in order to get into the South Harmon Institute of Technology?

6. Circle S_1 has radius 5. Circle S_2 has radius 7 and has its center lying on S_1. Circle S_3 has an integer radius and has its center lying on S_2. If the center of S_1 lies on S_3, how many possible values are there for the radius of S_3?

7. Jacob's analog clock has 12 equally spaced tick marks on the perimeter, but all the digits have been erased, so he doesn't know which tick mark corresponds to which hour. Jacob takes an arbitrary tick mark and measures clockwise to the hour hand and minute hand. He measures that the minute hand is 300 degrees clockwise of the tick mark, and that the hour hand is 70 degrees clockwise of the same tick mark. If it is currently morning, how many minutes past midnight is it?

8. How many ways are there to choose (not necessarily distinct) integers a, b, c from the set $\{1, 2, 3, 4\}$ such that $a^{(b^c)}$ is divisible by 4?

9. David has a collection of 40 rocks, 30 stones, 20 minerals and 10 gemstones. An operation consists of removing three objects, no two of the same type. What is the maximum number of operations he can possibly perform?

10. At certain store, a package of 3 apples and 12 oranges costs 5 dollars, and a package of 20 apples and 5 oranges costs 13 dollars. Given that apples and oranges can only be bought in these two packages, what is the minimum nonzero amount of dollars that must be spent to have an equal number of apples and oranges?

11. Let A, B, and C be distinct points on a line with $AB = AC = 1$. Square $ABDE$ and equilateral triangle ACF are drawn on the same side of line BC. What is the degree measure of the acute angle formed by lines EC and BF?

12. There are 25 ants on a number line; five at each of the coordinates 1, 2, 3, 4, and 5. Each minute, one ant moves from its current position to a position one unit away. What is the minimum number of minutes that must pass before it is possible for no two ants to be on the same coordinate?

13. There are three flies of negligible size that start at the same position on a circular track with circumference 1000 meters. They fly clockwise at speeds of 2, 6, and k meters per second, respectively, where k is some positive integer with $7 \le k \le 2013$. Suppose that at some point in time, all three flies meet at a location different from their starting point. How many possible values of k are there?

14. What is the smallest perfect square larger than 1 with a perfect square number of positive integer factors?

15. A permutation $a_1, a_2, \dots, a_{13}$ of the numbers from 1 to 13 is given such that $a_i > 5$ for $i = 1, 2, 3, 4, 5$. Determine the maximum possible value of

$$a_{a_1} + a_{a_2} + a_{a_3} + a_{a_4} + a_{a_5}.$$

16. Let S_1 and S_2 be two circles intersecting at points A and B. Let C and D be points on S_1 and S_2 respectively such that line CD is tangent to both circles and A is closer to line CD than B. If $\angle BCA = 52^\circ$ and $\angle BDA = 32^\circ$, determine the degree measure of $\angle CBD$.

17. Determine the number of ordered pairs of positive integers (x, y) with $y < x \le 100$ such that $x^2 - y^2$ and $x^3 - y^3$ are relatively prime. (Two numbers are *relatively prime* if they have no common factor other than 1.)

18. Determine the absolute value of the sum

$$\lfloor 2013 \sin 0^\circ \rfloor + \lfloor 2013 \sin 1^\circ \rfloor + \cdots + \lfloor 2013 \sin 359^\circ \rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

(You may use the fact that $\sin n^\circ$ is irrational for positive integers n not divisible by 30.)

19. A, B, C are points in the plane such that $\angle ABC = 90^\circ$. Circles with diameters BA and BC meet at D. If $BA = 20$ and $BC = 21$, then the length of segment BD can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. What is $m + n$?

20. Let $a_1, a_2, \dots, a_{2013}$ be a permutation of the numbers from 1 to 2013. Let $A_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$ for $n = 1, 2, \dots, 2013$. If the smallest possible difference between the largest and smallest values of $A_1, A_2, \dots, A_{2013}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

21. Dirock has a very neat rectangular backyard that can be represented as a 32×32 grid of unit squares. The rows and columns are each numbered $1, 2, \dots, 32$. Dirock is very fond of rocks, and places a rock in every grid square whose row and column number are both divisible by 3. Dirock would like to build a rectangular fence with vertices at the centers of grid squares and sides parallel to the sides of the yard such that

   a) The fence does not pass through any grid squares containing rocks;

   b) The interior of the fence contains exactly 5 rocks.

   In how many ways can this be done?

22. In triangle ABC, $AB = 28$, $AC = 36$, and $BC = 32$. Let D be the point on segment BC satisfying $\angle BAD = \angle DAC$, and let E be the unique point such that $DE \parallel AB$ and line AE is tangent to the circumcircle of ABC. Find the length of segment AE.

23. A set of 10 distinct integers S is chosen. Let M be the number of nonempty subsets of S whose elements have an even sum. What is the minimum possible value of M?

    Clarifications.
    - S is the "set of 10 distinct integers" from the first sentence.

24. For a permutation π of the integers from 1 to 10, define

$$S(\pi) = \sum_{i=1}^{9} (\pi(i) - \pi(i+1)) \cdot (4 + \pi(i) + \pi(i+1)),$$

    where $\pi(i)$ denotes the ith element of the permutation. Suppose that M is the maximum possible value of $S(\pi)$ over all permutations π of the integers from 1 to 10. Determine the number of permutations π for which $S(\pi) = M$.

25. Positive integers $x, y, z \leq 100$ satisfy

$$\begin{aligned} 1099x + 901y + 1110z &= 59800 \\ 109x + 991y + 101z &= 44556 \end{aligned}$$

    Compute $10000x + 100y + z$.

26. In triangle ABC, F is on segment AB such that CF bisects $\angle ACB$. Points D and E are on line CF such that lines AD, BE are perpendicular to CF. M is the midpoint of AB. If $ME = 13$, $AD = 15$, and $BE = 25$, find $AC + CB$.

27. Geodude wants to assign one of the integers $1, 2, 3, \ldots, 11$ to each lattice point (x, y, z) in a 3D Cartesian coordinate system. In how many ways can Geodude do this if for every lattice parallelogram $ABCD$, the positive difference between the sum of the numbers assigned to A and C and the sum of the numbers assigned to B and D must be a multiple of 11? (A *lattice point* is a point with all integer coordinates. A *lattice parallelogram* is a parallelogram with all four vertices lying on lattice points.)

    Clarifications.
    - The "positive difference" between two real numbers x and y is the quantity $|x - y|$.

28. Let S be the set of all lattice points (x, y) in the plane satisfying $|x| + |y| \leq 10$. Let $P_1, P_2, \ldots, P_{2013}$ be a sequence of 2013 (not necessarily distinct) points such that for every point Q in S, there exists at least one index i such that $1 \leq i \leq 2013$ and $P_i = Q$. Suppose that the minimum possible value of $|P_1P_2| + |P_2P_3| + \cdots + |P_{2012}P_{2013}|$ can be expressed in the form $a + b\sqrt{c}$, where a, b, c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$. (A *lattice point* is a point with all integer coordinates.)

    Clarifications.
    - $k = 2013$, i.e. the problem should read, "...there exists at least one index i such that $1 \leq i \leq 2013$ ...". An earlier version of the test read $1 \leq i \leq k$.

29. Let $\phi(n)$ denote the number of positive integers less than or equal to n that are relatively prime to n, and let $d(n)$ denote the number of positive integer divisors of n. For example, $\phi(6) = 2$ and $d(6) = 4$. Find the sum of all odd integers $n \leq 5000$ such that $n \mid \phi(n)d(n)$.

30. Pairwise distinct points $P_1, P_2, \ldots, P_{16}$ lie on the perimeter of a square with side length 4 centered at O such that $|P_iP_{i+1}| = 1$ for $i = 1, 2, \ldots, 16$. (We take P_{17} to be the point P_1.) We construct points $Q_1, Q_2, \ldots, Q_{16}$ as follows: for each i, a fair coin is flipped. If it lands heads, we define Q_i to be P_i; otherwise, we define Q_i to be the reflection of P_i over O. (So, it is possible for some of the Q_i to coincide.) Let D be the length of the vector $\overrightarrow{OQ_1} + \overrightarrow{OQ_2} + \cdots + \overrightarrow{OQ_{16}}$. Compute the expected value of D^2.

31. Beyond the Point of No Return is a large lake containing 2013 islands arranged at the vertices of a regular 2013-gon. Adjacent islands are joined with exactly two bridges. Christine starts on one of the islands with the intention of burning all the bridges. Each minute, if the island she is on has at least one bridge still joined to it, she randomly selects one such bridge, crosses it, and immediately burns it. Otherwise, she stops.

    If the probability Christine burns all the bridges before she stops can be written as $\frac{m}{n}$ for relatively prime positive integers m and n, find the remainder when $m+n$ is divided by 1000.

32. In $\triangle ABC$ with incenter I, $AB = 61$, $AC = 51$, and $BC = 71$. The circumcircles of triangles AIB and AIC meet line BC at points D ($D \neq B$) and E ($E \neq C$), respectively. Determine the length of segment DE.

33. Let n be a positive integer. E. Chen and E. Chen play a game on the n^2 points of an $n \times n$ lattice grid. They alternately mark points on the grid such that no player marks a point that is on or inside a non-degenerate triangle formed by three marked points. Each point can be marked only once. The game ends when no player can make a move, and the last player to make a move wins. Determine the number of values of n between 1 and 2013 (inclusive) for which the first player can guarantee a win, regardless of the moves that the second player makes.

34. For positive integers n, let $s(n)$ denote the sum of the squares of the positive integers less than or equal to n that are relatively prime to n. Find the greatest integer less than or equal to

    $$\sum_{n|2013} \frac{s(n)}{n^2},$$

    where the summation runs over all positive integers n dividing 2013.

35. The rows and columns of a 7×7 grid are each numbered $1, 2, \ldots, 7$. In how many ways can one choose 8 cells of this grid such that for every two chosen cells X and Y, either the positive difference of their row numbers is at least 3, or the positive difference of their column numbers is at least 3?

    Clarifications.

    - The "or" here is inclusive (as by convention, despite the "either"), i.e. X and Y are permitted if and only if they satisfy the row condition, the column condition, or both.

36. Let $ABCD$ be a nondegenerate isosceles trapezoid with integer side lengths such that $BC \parallel AD$ and $AB = BC = CD$. Given that the distance between the incenters of triangles ABD and ACD is $8!$, determine the number of possible lengths of segment AD.

37. Let M be a positive integer. At a party with 120 people, 30 wear red hats, 40 wear blue hats, and 50 wear green hats. Before the party begins, M pairs of people are friends. (Friendship is mutual.) Suppose also that no two friends wear the same colored hat to the party.

    During the party, X and Y can become friends if and only if the following two conditions hold:

    a) There exists a person Z such that X and Y are both friends with Z. (The friendship(s) between Z, X and Z, Y could have been formed during the party.)
    b) X and Y are not wearing the same colored hat.

    Suppose the party lasts long enough so that all possible friendships are formed. Let M_1 be the largest value of M such that regardless of which M pairs of people are friends before the party, there will always be at least one pair of people X and Y with different colored hats who are not friends after the party. Let M_2 be the smallest value of M such that regardless of which M pairs of people are friends before the party, every pair of people X and Y with different colored hats are friends after the party. Find $M_1 + M_2$.

Clarifications.

- The definition of M_2 should read, "Let M_2 be the *smallest* value of M such that...". An earlier version of the test read "largest value of M".

38. Triangle ABC has sides $AB = 25$, $BC = 30$, and $CA = 20$. Let P, Q be the points on segments AB, AC, respectively, such that $AP = 5$ and $AQ = 4$. Suppose lines BQ and CP intersect at R and the circumcircles of $\triangle BPR$ and $\triangle CQR$ intersect at a second point $S \neq R$. If the length of segment SA can be expressed in the form $\frac{m}{\sqrt{n}}$ for positive integers m, n, where n is not divisible by the square of any prime, find $m + n$.

39. Find the number of 8-digit base-6 positive integers $(a_1a_2a_3a_4a_5a_6a_7a_8)_6$ (with leading zeros permitted) such that $(a_1a_2\dots a_8)_6 \mid (a_{i+1}a_{i+2}\dots a_{i+8})_6$ for $i = 1, 2, \dots, 7$, where indices are taken modulo 8 (so $a_9 = a_1$, $a_{10} = a_2$, and so on).

40. Let ABC be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Let M be the midpoint of BC and let Γ be the circle passing through A and tangent to line BC at M. Let Γ intersect lines AB and AC at points D and E, respectively, and let N be the midpoint of DE. Suppose line MN intersects lines AB and AC at points P and O, respectively. If the ratio $MN : NO : OP$ can be written in the form $a : b : c$ with a, b, c positive integers satisfying $\gcd(a, b, c) = 1$, find $a + b + c$.

41. While there do not exist pairwise distinct real numbers a, b, c satisfying $a^2 + b^2 + c^2 = ab + bc + ca$, there do exist complex numbers with that property. Let a, b, c be complex numbers such that $a^2 + b^2 + c^2 = ab + bc + ca$ and $|a + b + c| = 21$. Given that $|a - b| = 2\sqrt{3}$, $|a| = 3\sqrt{3}$, compute $|b|^2 + |c|^2$.

Clarifications.

- The problem should read $|a + b + c| = 21$. An earlier version of the test read $|a + b + c| = 7$; that value is incorrect.
- $|b|^2 + |c|^2$ should be a positive integer, not a fraction; an earlier version of the test read "...for relatively prime positive integers m and n. Find $m + n$."

42. Find the remainder when
$$\prod_{i=0}^{100}(1 - i^2 + i^4)$$
is divided by 101.

43. In a tennis tournament, each competitor plays against every other competitor, and there are no draws. Call a group of four tennis players "ordered" if there is a clear winner and a clear loser (i.e., one person who beat the other three, and one person who lost to the other three.) Find the smallest integer n for which any tennis tournament with n people has a group of four tennis players that is ordered.

44. Suppose tetrahedron $PABC$ has volume 420 and satisfies $AB = 13$, $BC = 14$, and $CA = 15$. The minimum possible surface area of $PABC$ can be written as $m + n\sqrt{k}$, where m, n, k are positive integers and k is not divisible by the square of any prime. Compute $m + n + k$.

45. Let N denote the number of ordered 2011-tuples of positive integers $(a_1, a_2, \dots, a_{2011})$ with $1 \le a_1, a_2, \dots, a_{2011} \le 2011^2$ such that there exists a polynomial f of degree 4019 satisfying the following three properties:

- $f(n)$ is an integer for every integer n;
- $2011^2 \mid f(i) - a_i$ for $i = 1, 2, \dots, 2011$;
- $2011^2 \mid f(n + 2011) - f(n)$ for every integer n.

Find the remainder when N is divided by 1000.

46. Let ABC be a triangle with $\angle B - \angle C = 30^\circ$. Let D be the point where the A-excircle touches line BC, O the circumcenter of triangle ABC, and X, Y the intersections of the altitude from A with the incircle with X in between A and Y. Suppose points A, O and D are collinear. If the ratio $\frac{AO}{AX}$ can be expressed in the form $\frac{a+b\sqrt{c}}{d}$ for positive integers a, b, c, d with $\gcd(a, b, d) = 1$ and c not divisible by the square of any prime, find $a + b + c + d$.

47. Let $f(x, y)$ be a function from ordered pairs of positive integers to real numbers such that

$$f(1, x) = f(x, 1) = \frac{1}{x} \quad \text{and} \quad f(x+1, y+1)f(x, y) - f(x, y+1)f(x+1, y) = 1$$

for all ordered pairs of positive integers (x, y). If $f(100, 100) = \frac{m}{n}$ for two relatively prime positive integers m, n, compute $m + n$.

48. ω is a complex number such that $\omega^{2013} = 1$ and $\omega^m \neq 1$ for $m = 1, 2, \ldots, 2012$. Find the number of ordered pairs of integers (a, b) with $1 \leq a, b \leq 2013$ such that

$$\frac{(1 + \omega + \cdots + \omega^a)(1 + \omega + \cdots + \omega^b)}{3}$$

is the root of some polynomial with integer coefficients and leading coefficient 1. (Such complex numbers are called *algebraic integers*.)

49. In $\triangle ABC$, $CA = 1960\sqrt{2}$, $CB = 6720$, and $\angle C = 45^\circ$. Let K, L, M lie on BC, CA, and AB such that $AK \perp BC$, $BL \perp CA$, and $AM = BM$. Let N, O, P lie on KL, BA, and BL such that $AN = KN$, $BO = CO$, and A lies on line NP. If H is the orthocenter of $\triangle MOP$, compute HK^2.

Clarifications.

- Without further qualification, "XY" denotes line XY.

50. Let S denote the set of words $W = w_1 w_2 \ldots w_n$ of any length $n \geq 0$ (including the empty string λ), with each letter w_i from the set $\{x, y, z\}$. Call two words U, V *similar* if we can insert a string $s \in \{xyz, yzx, zxy\}$ of three consecutive letters somewhere in U (possibly at one of the ends) to obtain V or somewhere in V (again, possibly at one of the ends) to obtain U, and say a word W is *trivial* if for some nonnegative integer m, there exists a sequence $W_0, W_1, \ldots, W_m$ such that $W_0 = \lambda$ is the empty string, $W_m = W$, and W_i, W_{i+1} are similar for $i = 0, 1, \ldots, m-1$. Given that for two relatively prime positive integers p, q we have

$$\frac{p}{q} = \sum_{n \geq 0} f(n) \left(\frac{225}{8192} \right)^n,$$

where $f(n)$ denotes the number of trivial words in S of length $3n$ (in particular, $f(0) = 1$), find $p+q$.