The Online Math Open Solutions
January 16-23, 2012

Online Math Open

Note: Some of the solutions were taken from the Art of Problem Solving discussion threads. We have given the posters due credit at the ends of the solutions.

1. The average of two positive real numbers is equal to their difference. What is the ratio of the larger number to the smaller one?

   Solution. The answer is $\boxed{3}$.

   Let the numbers be x and y and assume without loss of generality $x > y$. Then we have $\frac{x+y}{2} = x - y$. It follows that $x + y = 2x - 2y$, so $x = 3y$ and the answer is 3.

   This problem was proposed by Ray Li.

2. How many ways are there to arrange the letters A, A, A, H, H in a row so that the sequence HA appears at least once?

   Solution. The answer is $\boxed{9}$.

   Every sequence is legal except $AAAHH$. Thus there are $\binom{5}{2} - 1 = 9$ sequences of letters.

   This problem was proposed by Ray Li.

3. A lucky number is a positive number whose digits are only 4 or 7. What is the 17th smallest lucky number?

   Clarifications:

   - Lucky numbers are positive.
   - "only 4 or 7" includes combinations of 4 and 7, as well as only 4 and only 7. That is, 4 and 47 are both lucky numbers.

   Solution. The answer is $\boxed{4474}$.

   There are 2 1-digit lucky numbers, 4 2-digit lucky numbers, and 8 3-digit lucky numbers. Thus we want the 3rd smallest 4-digit lucky number. We list them: 4444, 4447, 4474, 4477, ... so our answer is 4474.

   This problem was proposed by Ray Li. The solution was provided by ksun48.

4. How many positive even numbers have an even number of digits and are less than 10000?

   Solution. The answer is $\boxed{4545}$.

   Exactly half of the 90 2-digit numbers and exactly half of the 9000 4-digit numbers are even, giving a total of $\frac{90+9000}{2} = 4545$ numbers.

   This problem was proposed by Ray Li.

5. Congruent circles Γ_1 and Γ_2 have radius 2012, and the center of Γ_1 lies on Γ_2. Suppose that Γ_1 and Γ_2 intersect at A and B. The line through A perpendicular to AB meets Γ_1 and Γ_2 again at C and D, respectively. Find the length of CD.

   Solution. The answer is $\boxed{4024}$.

   The center of Γ_2, which we denote O_2, must lie on Γ_1, which we denote O_1, because their centers are 2012 apart. Also, $\angle CO_1A = 60°$, because $\angle O_2O_1A = \angle O_2O_1B = 60°$. Similarly, $\angle DO_2A = 60°$. Thus $\triangle CO_1A$ and $\triangle DO_2A$ are equilateral. This implies that $CA = DA = 2012$, so $CD = 4024$.

   This problem was proposed by Ray Li. The solution was provided by ksun48.

6. Alice's favorite number has the following properties:

   - It has 8 distinct digits.

- The digits are decreasing when read from left to right.
- It is divisible by 180.

What is Alice's favorite number?

Solution. The answer is $\boxed{97654320}$.

Because Alice's favorite number is divisible by 180, it must end in a 0. It must also be divisible by 20, so the penultimate digit must be even. The number has decreasing distinct digits, so it must be a 2. Finally, the number must be divisible by 9, which means the sum of the digits must be divisible by 9. The number must contain all the digits from 0 to 9 exactly once except 2 of them, and it is already missing 1. The remaining sum must be 36, so the other missing digit is 8. Thus our integer is 97654320.

This problem was proposed by Anderson Wang. The solution was provided by ksun48.

7. A board 64 inches long and 4 inches high is inclined so that the long side of the board makes a 30 degree angle with the ground. The distance from the highest point on the board to the ground can be expressed in the form $a + b\sqrt{c}$ where a, b, c are positive integers and c is not divisible by the square of any prime. What is $a + b + c$?

   Clarifications:

   - The problem is intended to be a two-dimensional problem. The board's dimensions are 64 by 4. The long side of the board makes a 30 degree angle with the ground. One corner of the board is touching the ground.

   Solution. The answer is $\boxed{37}$.

   Call the vertices of the board A, B, C, D where A is the vertex on the ground and AB is the long side of the board. The height of point B, by 30-60-90 triangles is $64 \cdot \frac{1}{2}$, and the difference in heights of C and B is, again by 30-60-90 triangles is $4 \cdot \frac{\sqrt{3}}{2}$. This gives that C has a height of $32 + 2\sqrt{3}$, so the answer is $32 + 2 + 3 = 37$.

   This problem was proposed by Ray Li.

8. An $8 \times 8 \times 8$ cube is painted red on 3 faces and blue on 3 faces such that no corner is surrounded by three faces of the same color. The cube is then cut into 512 unit cubes. How many of these cubes contain both red and blue paint on at least one of their faces?

   Clarifications:

   - The problem asks for the number of cubes that contain red paint on at least one face and blue paint on at least one other face, not for the number of cubes that have both colors of paint on at least one face (which can't even happen).

   Solution. The answer is $\boxed{56}$.

   The blue faces and red faces meet at 8 edges. Each edge has 8 cubes. The 8 vertices are each counted twice, so our final answer is $8 \cdot 8 - 8 = 56$.

   This problem was proposed by Ray Li. The solution was provided by KingSmasher3.

9. At a certain grocery store, cookies may be bought in boxes of 10 or 21. What is the minimum positive number of cookies that must be bought so that the cookies may be split evenly among 13 people?

   Solution. The answer is $\boxed{52}$.

   The number of cookies must be a multiple of 3. It is easy to see that $13, 26, 39$ are all impossible, but $52 = 2 \times 21 + 10$, so the answer is 52.

   This problem was proposed by Ray Li.

10. A drawer has 5 pairs of socks. Three socks are chosen at random. If the probability that there is a pair among the three is $\frac{m}{n}$, where m and n are relatively prime positive integers, what is $m+n$?

    Solution. The answer is $\boxed{4}$.

    There are 5 ways to chose a pair, then 8 ways to chose the remaining sock, giving a probability of

    $$\frac{5\cdot 8}{\binom{10}{3}}=\frac{1}{3},$$

    so the answer is $1+3=4$.

    This problem was proposed by Ray Li.

11. If

    $$\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{4x^3}+\frac{1}{8x^4}+\frac{1}{16x^5}+\cdots=\frac{1}{64},$$

    and x can be expressed in the form $\frac{m}{n}$, where m,n are relatively prime positive integers, find $m+n$.

    Solution. The answer is $\boxed{131}$.

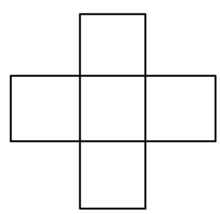
    This is simply a geometric series:

    $$\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{4x^3}+\frac{1}{8x^4}+\frac{1}{16x^5}+\cdots=\frac{1/x}{1-1/2x}=\frac{1}{64}\implies\frac{2}{2x-1}=\frac{1}{64}.$$

    Solving, we find $128=2x-1\implies x=\frac{129}{2}$, so the answer is $129+2=131$.

    This problem was proposed by Ray Li. The solution was provided by tc1729.

12. A *cross-pentomino* is a shape that consists of a unit square and four other unit squares each sharing a different edge with the first square. If a cross-pentomino is inscribed in a circle of radius R, what is $100R^2$?

    Solution. The answer is $\boxed{250}$.

    By the Pythagorean theorem, the distance from center to any corner is $\sqrt{\frac{1}{2}^2+\frac{3}{2}^2}=\sqrt{\frac{5}{2}}$, so the answer is $100\cdot\frac{5}{2}=250$.

    This problem was proposed by Ray Li. The solution was provided by BarbieRocks.

13. A circle ω has center O and radius r. A chord BC of ω also has length r, and the tangents to ω at B and C meet at A. Ray AO meets ω at D past O, and ray OA meets the circle centered at A with radius AB at E past A. Compute the degree measure of $\angle DBE$.

    Solution. The answer is $\boxed{135}$.

    Our strategy is to first find $\angle BDE$ and $\angle BED$, then subtract their sum from 180 to get the desired answer.

    Note that $\triangle BOC$ is equilateral, so $\angle BOC=60$ and $\angle BOA=30$. By inscribed angles, angle BDE is half this, or 15 degrees. Now examine $\triangle ABO$. $\angle ABO$ is right since AB is a tangent, and $\angle BOA=30$, so $\angle BAO=60$. Therefore, $\angle BED=30$ by inscribed angles, and the answer is $180-30-15=135$ degrees.

    This problem was proposed by Ray Li. The solution was provided by professordad.

14. Al told Bob that he was thinking of 2011 distinct positive integers. He also told Bob the sum of those integers. From this information, Bob was able to determine all 2011 integers. How many possible sums could Al have told Bob?

    Solution. The answer is $\boxed{2}$.

    Let $s = 1 + 2 + \cdots + 2010 + 2011$. If the sum S told is larger than $s + 1$, then the integers could be either $1, 2, \ldots, 2009, 2010, 2011 + S - s$, or $1, 2, \ldots, 2009, 2011, 2010 + S - s$. Clearly the sum is at least s, so the only possibilities are s and $s + 1$, and the answer is 2.

    This problem was proposed by Ray Li. The solution was provided by mavropnevma.

15. Five bricklayers working together finish a job in 3 hours. Working alone, each bricklayer takes at most 36 hours to finish the job. What is the smallest number of minutes it could take the fastest bricklayer to complete the job alone?

    Solution. The answer is $\boxed{270}$.

    Let the time of the fastest bricklayer be t. To minimize t, let the other 4 workers take 36 hours each. Now let the amount of work be 1 (whatever unit). By the $D = RT$ formula, we have

    $$\frac{1}{\frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{t}} = 3,$$

    whence $\frac{1}{t} + \frac{1}{9} = \frac{1}{3}$, and $t = \frac{9}{2}$ hours or 270 minutes.

    This problem was proposed by Ray Li. The solution was provided by professordad.

16. Let $A_1B_1C_1D_1A_2B_2C_2D_2$ be a unit cube, with $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ opposite square faces, and let M be the center of face $A_2B_2C_2D_2$. Rectangular pyramid $MA_1B_1C_1D_1$ is cut out of the cube. If the surface area of the remaining solid can be expressed in the form $a + \sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime, find $a + b$.

    Solution. The answer is $\boxed{10}$.

    The solid's surface area is 5 unit squares (5 square units), and 4 isosceles triangles. The base of each triangle is just 1, and the height is $\sqrt{(1/2)^2 + (1)^2} = \sqrt{5}/2$, so the area of each triangle is $\sqrt{5}/4$, and 4 triangles, for an area of the triangles is $\sqrt{5}$. The total area is $5 + \sqrt{5}$, so $a + b = 10$.

    This problem was proposed by Alex Zhu.

17. Each pair of vertices of a regular 10-sided polygon is connected by a line segment. How many unordered pairs of distinct parallel line segments can be chosen from these segments?

    Solution. The answer is $\boxed{80}$.

    There are two possibilities: either the diagonals face a side (i.e. are parallel to some side), or the diagonals face a vertex. For each case, there are 5 possible directions (1 for each pair of vertices and sides). In the first case, there are 5 diagonals for each direction, giving $\binom{5}{2} = 10$ pairs for each direction. In the second case there are 4 diagonals for each direction, giving $\binom{4}{2} = 6$ pairs for each direction. This gives $5 \times 10 + 5 \times 6 = 80$ total pairs.

    This problem was proposed by Ray Li.

18. The sum of the squares of three positive numbers is 160. One of the numbers is equal to the sum of the other two. The difference between the smaller two numbers is 4. What is the positive difference between the cubes of the smaller two numbers?

    Clarifications:

    - The problem should ask for the positive difference.

    Solution. The answer is $\boxed{320}$.

Let the largest of the three be a. We have

$$a^2 + b^2 + c^2 = 160$$
$$a = b + c$$
$$b - c = 4$$

Plugging in $a = b+c$ into the first equation gives $2b^2+2bc+2c^2 = 160$, so $b^2+bc+c^2 = 80$. Multiplying this by $b - c = 4$ gives $b^3 - c^3 = 320$.

This problem was proposed by Ray Li. The solution was provided by professordad.

19. There are 20 geese numbered 1 through 20 standing in a line. The even numbered geese are standing at the front in the order $2, 4, \ldots, 20$, where 2 is at the front of the line. Then the odd numbered geese are standing behind them in the order, $1, 3, 5, \ldots, 19$, where 19 is at the end of the line. The geese want to rearrange themselves in order, so that they are ordered $1, 2, \ldots, 20$ (1 is at the front), and they do this by successively swapping two adjacent geese. What is the minimum number of swaps required to achieve this formation?

Solution. The answer is $\boxed{55}$.

We can successively swap $1, 3, 5, \ldots, 19$ into their proper positions. This takes 10 swaps for 1, then 9 swaps for 3, and so on, giving $10 + 9 + 8 + \cdots + 1 = 55$ total swaps. To see that 55 is the minimum, notice that initially the number of pairs of geese (a, b) such that $a < b$ but geese a is after geese b is 55, (one can use a similar counting argument to above) and the number of there bad pairs decreases by at most 1 with each swap. Because when the geese are ordered, there are 0 such bad pairs, we need at least 55 swaps, so the answer is 55.

Note: In general, for any permutation π of a sequence of integers, the number of pairs (a, b) such that $a < b$ and a comes after b is known as the number of *inversions* of the permutation.

This problem was proposed by Ray Li.

20. Let ABC be a right triangle with a right angle at C. Two lines, one parallel to AC and the other parallel to BC, intersect on the hypotenuse AB. The lines cut the triangle into two triangles and a rectangle. The two triangles have areas 512 and 32. What is the area of the rectangle?

Solution. The answer is $\boxed{256}$.

Let the rectangle be $PQRC$, where P is on AC, Q is on AB, and R is on BC. Let $AP = a$, $PQ = b$, $QR = c$, and $BR = d$. From similar triangles, we have $ad = bc$. Also, $cd = 64$, and $ab = 1024$ from the given information. Multiplying these gives $abcd = 1024 \cdot 64$. Since $ad = bc$, $ad = bc = 32 \cdot 8 = 256$.

This problem was proposed by Ray Li. The solution was provided by professordad.

21. If

$$2011^{2011^{2012}} = x^x$$

for some positive integer x, how many positive integer factors does x have?

Solution. The answer is $\boxed{2012}$.

Notice that $x = 2011^{2011}$. Because 2011 is prime, x has 2012 positive integer factors.

This problem was proposed by Alex Zhu.

22. Find the largest prime number p such that when 2012! is written in base p, it has at least p trailing zeroes.

Solution. The answer is $\boxed{43}$.

The number of trailing zeros when 2012! is written in base p is given by

$$\lfloor \frac{2012}{p} \rfloor + \lfloor \frac{2012}{p^2} \rfloor + \cdots > p.$$

Thus, we have

$$\frac{2012}{p-1} = \frac{2012}{p} + \frac{2012}{p^2} + \cdots > \lfloor \frac{2012}{p} \rfloor + \lfloor \frac{2012}{p^2} \rfloor + \cdots > p,$$

whence $p(p-1) < 2012$. It follows that because p is a prime $p \leq 43$. It is easy to check that $p = 43$ satisfies the originally inequality, so 43 is the answer.

This problem was proposed by Alex Zhu.

23. Let ABC be an equilateral triangle with side length 1. This triangle is rotated by some angle about its center to form triangle DEF. The intersection of ABC and DEF is an equilateral hexagon with an area that is $\frac{4}{5}$ the area of ABC. The side length of this hexagon can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. What is $m + n$?

    Solution. The answer is $\boxed{7}$.

    Let O be the center of the two triangles. Let the hexagon be $PQRSTU$ with P, Q on side AB. Notice that O is equidistant from every side of the hexagon, so the areas of triangles $OPQ, OQR, ORS, \ldots$ are all equal. In particular, they are all equal to $\frac{1}{6} \cdot \frac{4}{5} = \frac{2}{15}$ of the area of ABC. Let s be the length of side PQ, and let h be the height of ABC. Because the height of OPQ is $\frac{h}{3}$, we have

    $$\frac{[OPQ]}{[ABC]} = \frac{s \cdot \frac{h}{3}}{1 \cdot h} = \frac{2}{15}$$

    so it follows that $s = \frac{2}{5}$ and the answer is $2 + 5 = 7$.

    Note: Here $[XYZ]$ denotes the area of triangle XYZ.

    This problem was proposed by Ray Li.

24. Find the number of ordered pairs of positive integers (a, b) with $a + b$ prime, $1 \leq a, b \leq 100$, and $\frac{ab+1}{a+b}$ is an integer.

    Solution. The answer is $\boxed{51}$.

    Notice $\frac{ab+1}{a+b} + 1 = \frac{ab+a+b+1}{a+b} = \frac{(a+1)(b+1)}{a+b}$ must be an integer, whence $a + b | a + 1$ or $a + b | b + 1$, so $a = 1$ or $b = 1$. It follows that we need $a + 1$ or $b + 1$ to be a prime. Since there are 26 primes less than or equal to 101, (the maximum possible value of $a + 1$) we obtain $26 \times 2 = 52$ pairs (a, b), but we overcount the $a = b = 1$ case, so the answer is $\boxed{51}$.

    This problem was proposed by Alex Zhu.

25. Let a, b, c be the roots of the cubic $x^3 + 3x^2 + 5x + 7$. Given that P is a cubic polynomial such that $P(a) = b + c$, $P(b) = c + a$, $P(c) = a + b$, and $P(a + b + c) = -16$, find $P(0)$.

    Solution. The answer is $\boxed{11}$.

    Notice that $a + b + c = -3$ by Vieta's. Let $Q(x)$ be a polynomial such that $Q(x) = P(x) + x + 3$. Notice that $Q(x)$ has a, b, c as roots by the first three conditions, so $Q(x) = c(x^3 + 3x^2 + 5x + 7)$ for some constant c. By the last condition, we find $c(-27 + 27 - 15 + 7) = -16$, so $c = 2$, and $Q(x) = 2x^3 + 6x^2 + 10x + 14$. Now we have $14 = Q(0) = P(0) + 0 + 3$, so $P(0) = 11$.

    This problem was proposed by Alex Zhu.

26. Xavier takes a permutation of the numbers 1 through 2011 at random, where each permutation has an equal probability of being selected. He then cuts the permutation into increasing contiguous subsequences, such that each subsequence is as long as possible. Compute the expected number of such subsequences.

    Clarifications:

    - An increasing contiguous subsequence is an increasing subsequence all of whose terms are adjacent in the original sequence. For example, 1,3,4,5,2 has two maximal increasing contiguous subsequences: (1,3,4,5) and (2).

Solution. The answer is $\boxed{1006}$.

When every permutation with n subsequences is written backwards, the number of subsequences in the new permutation is $2012 - n$. Thus, every pair averages to our answer, 1006.

This problem was proposed by Alex Zhu. The solution was provided by Draco.

27. Let a and b be real numbers that satisfy

$$a^4 + a^2b^2 + b^4 = 900,$$

$$a^2 + ab + b^2 = 45.$$

Find the value of $2ab$.

Solution. The answer is $\boxed{25}$.

Dividing the two equations, we obtain $a^2 - ab + b^2 = 20$. Subtracting this from $a^2 + ab + b^2 = 45$, we obtain $2ab = 25$ as desired.

This problem was proposed by Ray Li.

28. A fly is being chased by three spiders on the edges of a regular octahedron. The fly has a speed of 50 meters per second, while each of the spiders has a speed of r meters per second. The spiders choose the (distinct) starting positions of all the bugs, with the requirement that the fly must begin at a vertex. Each bug knows the position of each other bug at all times, and the goal of the spiders is for at least one of them to catch the fly. What is the maximum c so that for any $r < c$, the fly can always avoid being caught?

Solution. The answer is $\boxed{25}$.

Suppose that $r < \frac{50}{2} = 25$. If the fly is on a vertex of the octahedron, then in the time it would take for the fly to crawl directly to one of the four adjacent vertices, each spider could possibly reach at most one vertex, thus stopping the fly from getting there. A spider can also block the fly by being on an edge adjacent to the fly's vertex, but again the spider blocks only one edge. (The circles in the picture represent the range of motion of the spiders... kind of.) Because there are only three spiders, the fly can always run to another vertex. This situation repeats indefinitely, so the fly is never caught.

If $r > 25$, then two spiders at the midpoints of opposite outer edges can contain the fly, each blocking two vertices. The third spider can pick of the fly at his leisure by making its way to the central vertex, cornering the fly on one of the four central edges. If $r = 25$, then the fly cannot definitely escape, though I'm not sure it's guaranteed it gets caught, either. Regardless, our answer is $c = 25$.

This problem was proposed by Anderson Wang. The solution was provided by ziv.

29. How many positive integers a with $a \le 154$ are there such that the coefficient of x^a in the expansion of

$$(1 + x^7 + x^{14} + \cdots + x^{77})(1 + x^{11} + x^{22} + \cdots + x^{77})$$

is zero?

Solution. The answer is $\boxed{60}$.

This is the same as finding the number of nonegative integers a since the coefficient of x^0 is nonzero. We claim that all the nonzero coefficients are 1, except for the coefficient of 77, which is 2. Notice that this is equivalent to finding the number of positive integers a that are not expressible in the form $7a + 11b$ with $a \le 11, b \le 7$. If $7a + 11b = 7c + 11d$, constrained to $0 \le a, c \le 11$ and $0 \le b, d \le 7$, then either $7a + 11b = 77$ or $(a, b) = (c, d)$ showing that 77 is the only coefficient that appears more than once.

Now, because the sum of the coefficients is $8 \cdot 12 = 96$ (plug in $x = 1$,) we have $96 - 1$ distinct nonzero coefficients. This gives us $155 - 95 = 60$ values of a with $[x^a] = 0$.

This problem was proposed by Ray Li.

30. The Lattice Point Jumping Frog jumps between lattice points in a coordinate plane that are exactly 1 unit apart. The Lattice Point Jumping Frog starts at the origin and makes 8 jumps, ending at the origin. Additionally, it never lands on a point other than the origin more than once. How many possible paths could the frog have taken?

Clarifications:

- The Lattice Jumping Frog is allowed to visit the origin more than twice.
- The path of the Lattice Jumping Frog is an ordered path, that is, the order in which the Lattice Jumping Frog performs its jumps matters.

Solution. The answer is $\boxed{280}$.

Call a k-string a string of moves that start and end with the origin, that don't go through the origin other than at its endpoints.

If the path is an 8-string, then you trace out a polyomino. There are 3 polynominoes in this case: A 2×2, square, a straight segment with 3 unit squares, and an L shape. The square can be rotated in 1 way, the straight segment in 2 ways, and the L in 4 ways. For each polyomino and orientation, the origin of the coordinate system can be placed at 8 positions on the boundary, and given the polyomino, orientation, and origin location, there are 2 ways to traverse the path. This gives $2 \times 8 \times (1+2+4) = 112$ paths in this case.

If the path has a 6-string, then the path is a dangling line segment attached to a polyomino. This case must be a domino with a dangling segment. The domino has 2 possible orientations. For each orientation there are 6 places to put the origin. If the origin is at a vertex, there are 4 places it could go, and the dangling segment has 2 possible directions. There are 2 ways to choose the traversal order (segment first or domino first), and there are 2 ways to traverse the domino. Thus there are $4 \times 2 \times 2 \times 2 = 32$ paths in this case. If the origin is not at a vertex, there are 2 places it could go, and the dangling segment has 1 possible direction. Once again there are 2 ways to choose traversal order and 2 ways to traverse the domino, giving $2 \times 2 \times 2 = 8$ paths in this case. Because there are 2 orientations of the domino, we have $2 \times (32 + 8) = 80$ paths in this case.

If the path only has 4-strings and 2-strings, there are three cases: (1) Two 4-strings. This is simply two unit squares attached to each other at a corner. There are two ways to choose the orientation of the squares, and 8 ways to traverse each orientation, for 16; (2) One 4-string and two 2-strings. This is just two dangling line segments attached to one corner of a unit square. There are 4 orientations, 2 ways to traverse the square, $3! = 6$ ways to choose the order of the square and segments. This gives $4 \cdot 2 \cdot 6 = 48$; (3) Four 2-strings. This is just a star. There are $4! = 24$ ways to choose the order of the stars.

This gives a total of $112 + 80 + 16 + 48 + 24 = 280$ paths.

Author's comment: I never really intended for this problem to be this ugly. As it was in the test, the problem statement clearly implied that the frog could visit the origin more than twice, but I only intended the problem to be as hard as the first case.

This problem was proposed by Ray Li.

31. Let ABC be a triangle inscribed in circle Γ, centered at O with radius 333. Let M be the midpoint of AB, N be the midpoint of AC, and D be the point where line AO intersects BC. Given that lines MN and BO concur on Γ and that $BC = 665$, find the length of segment AD.

Solution. The answer is $\boxed{444}$.

Let AD intersect MN at Q, and let MN intersect BO at P. Since $BO = OP$ and $MN \parallel BC$, $QO = OD$. We also have $AQ = QD$, so $AD = \frac{4}{3}AO = 444$. The length of BC isn't needed!

This problem was proposed by Alex Zhu. The solution was provided by ftong.

32. The sequence $\{a_n\}$ satisfies $a_0 = 201, a_1 = 2011$, and $a_n = 2a_{n-1} + a_{n-2}$ for all $n \geq 2$. Let

$$S = \sum_{i=1}^{\infty} \frac{a_{i-1}}{a_i^2 - a_{i-1}^2}.$$

What is $\frac{1}{S}$?

Solution. The answer is $\boxed{3620}$.

Using partial fractions,

$$\frac{a_{i-1}}{a_i^2 - a_{i-1}^2} = \frac{1}{2}\left(\frac{1}{a_i - a_{i-1}} - \frac{1}{a_i + a_{i-1}}\right).$$

By the recurrence relation, $a_i + a_{i-1} = a_{i+1} - a_i$. So we get

$$S = \sum_{i=1}^{\infty} \frac{1}{2}\left(\frac{1}{a_i - a_{i-1}} - \frac{1}{a_{i+1} - a_i}\right) = \frac{1}{2(a_1 - a_0)} = \frac{1}{3620}.$$

So the answer is 3620.

This problem was proposed by Ray Li. The solution was provided by ftong.

33. You are playing a game in which you have 3 envelopes, each containing a uniformly random amount of money between 0 and 1000 dollars. (That is, for any real $0 \le a < b \le 1000$, the probability that the amount of money in a given envelope is between a and b is $\frac{b-a}{1000}$.) At any step, you take an envelope and look at its contents. You may choose either to keep the envelope, at which point you finish, or discard it and repeat the process with one less envelope. If you play to optimize your expected winnings, your expected winnings will be E. What is $\lfloor E \rfloor$, the greatest integer less than or equal to E?

Solution. The answer is $\boxed{695}$.

For one envelope, clearly your expected winnings is 500. For two envelopes, open it. If it has over 500 you should keep it, otherwise throw it out. Hence your expected winning should be $\frac{1}{2}\cdot 750 + \frac{1}{2}\cdot 500 = 625$ Let's say you have three envelopes. If the first envelope has over 625 you should keep it, otherwise throw it out. Thus the expected winning is $\frac{3}{8}\cdot(625+1000)/2 + \frac{5}{8}\cdot 625 = \frac{11125}{16} = 695 + \frac{5}{16}$, hence $\lfloor E \rfloor = 695$.

This problem was proposed by Alex Zhu. The solution was provided by dinoboy.

34. Let p, q, r be real numbers satisfying

$$\frac{(p+q)(q+r)(r+p)}{pqr} = 24$$

$$\frac{(p-2q)(q-2r)(r-2p)}{pqr} = 10.$$

Given that $\frac{p}{q} + \frac{q}{r} + \frac{r}{p}$ can be expressed in the form $\frac{m}{n}$, where m, n are relatively prime positive integers, compute $m + n$.

Solution. The answer is $\boxed{67}$.

Let $u = \frac{p}{q}$, $v = \frac{q}{r}$, and $w = \frac{r}{p}$. Clearly, $uvw = 1$. We seek $u + v + w$. Note that the two equations can be rewritten as

$$\frac{(p+q)(q+r)(r+p)}{pqr} = \left(\frac{p+q}{q}\right)\left(\frac{q+r}{r}\right)\left(\frac{r+p}{p}\right) = (u+1)(v+1)(w+1) = 24,$$

and

$$\frac{(p-2q)(q-2r)(r-2p)}{pqr} = \left(\frac{p-2q}{q}\right)\left(\frac{q-2r}{r}\right)\left(\frac{r-2p}{p}\right) = (u-2)(v-2)(w-2) = 10.$$

Expanding, we get $uvw+uv+vw+wu+u+v+w+1 = 24$ and $uvw-2(uv+vw+wu)+4(u+v+w)-8 = 10$. Adding twice the first equation to the second, we get $3uvw + 6(u+v+w) - 6 = 58$, so $u + v + w = \frac{58+6-3}{6} = \frac{61}{6}$, giving an answer of $61 + 6 = 67$.

This problem was proposed by Alex Zhu.

35. Let $s(n)$ be the number of 1's in the binary representation of n. Find the number of ordered pairs of integers (a, b) with $0 \le a < 64, 0 \le b < 64$ and $s(a+b) = s(a) + s(b) - 1$.

Solution. The answer is $\boxed{648}$.

The condition means there is exactly 1 place where a and b both have a 1 in the same place value spot, but both have 0's in the next (left) bit. a and b must both have 6 bits (leading 0's allowed). Thus the number of ways to do this is 3^5 (if the 1's are both in the 6th bit) + $5 \cdot 3^4$ (the 1 pair are in the other bits), so 648 total.

This problem was proposed by Anderson Wang. The solution was provided by ksun48.

36. Let s_n be the number of ordered solutions to $a_1 + a_2 + a_3 + a_4 + b_1 + b_2 = n$, where a_1, a_2, a_3 and a_4 are elements of the set $\{2, 3, 5, 7\}$ and b_1 and b_2 are elements of the set $\{1, 2, 3, 4\}$. Find the number of n for which s_n is odd.

Clarifications:

- s_n is the number of *ordered* solutions $(a_1, a_2, a_3, a_4, b_1, b_2)$ to the equation, where each a_i lies in $\{2, 3, 5, 7\}$ and each b_i lies in $\{1, 2, 3, 4\}$.

Solution. The answer is $\boxed{12}$.

We just want the number of odd coefficients in the generating function $(x^2+x^3+x^5+x^7)^4(x^1+x^2+x^3+x^4)^2$. But it is well-known that $f(x)^{2^k} - f(x^{2^k})$ has all even coefficients for any $f \in \mathbb{Z}[x]$ and positive integer k (prove it!), so this is also the number of odd coefficients in $(x^8+x^{12}+x^{20}+x^{28})(x^2+x^4+x^6+x^8)$. (Why?)

The rest is straightforward computation:

$$(x^8+x^{12}+x^{20}+x^{28})(x^2+x^4+x^6+x^8) = x^{10}+x^{12}+2x^{14}+2x^{16}+x^{20}+x^{22}+x^{24}+x^{26}+x^{28}+x^{30}+x^{32}+x^{34}+x^{36},$$

which has 12 odd coefficients.

This problem was proposed by Alex Zhu.

37. In triangle ABC, $AB = 1$ and $AC = 2$. Suppose there exists a point P in the interior of triangle ABC such that $\angle PBC = 70^\circ$, and that there are points E and D on segments AB and AC, such that $\angle BPE = \angle EPA = 75^\circ$ and $\angle APD = \angle DPC = 60^\circ$. Let BD meet CE at Q, and let AQ meet BC at F. If M is the midpoint of BC, compute the degree measure of $\angle MPF$.

Solution. The answer is $\boxed{25}$.

By Ceva's,

$$\frac{BE}{EA} \cdot \frac{AD}{DC} \cdot \frac{CF}{FB} = 1.$$

By angle bisector theorem,

$$\frac{BE}{EA} \cdot \frac{AD}{DC} = \frac{BP}{PC},$$

so $\angle FPC = \angle BPC/2 = 45^\circ$. Since $\triangle BPC$ is right, $\angle MPC = \angle MCP = 90^\circ - \angle PBF = 20^\circ$. Therefore, $\angle MPF = \angle FPC - \angle MPC = 25^\circ$.

This problem was proposed by Alex Zhu and Ray Li. The solution was provided by ftong.

38. Let S denote the sum of the 2011th powers of the roots of the polynomial $(x - 2^0)(x - 2^1) \cdots (x - 2^{2010}) - 1$. How many 1's are in the binary expansion of S?

Solution. The answer is $\boxed{2018}$.

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$. Newton's sums tell us

$$\begin{aligned} a_n S_1 + a_{n-1} &= 0 \\ a_n S_2 + a_{n-1} S_1 + 2a_{n-2} &= 0 \\ &\vdots \\ a_n S_n + \cdots + a_1 S_1 + n a_0 &= 0, \end{aligned}$$

where S_k is the sum of the kth power of the roots.

Now, suppose we subtract 1 from a_0. The first $(n-1)$ sums do not change, as they are not dependent on a_0. The last S_n must increase by n/a_n.

Application to this problem: The sum of the 2011th roots is now $1+2^{2011}+2^{4022}+\cdots+2^{2010\cdot 2011}+2011$. Now, 2012 in binary is $4+8+16+64+128+256+512+1024 = 11111011100_2$, and the sum of the remaining terms is $1000000\cdots10000000\cdots$ (where there are 2010 ones) in binary. So the digit sum is $2010+8 = 2018$.

This problem was proposed by Alex Zhu. The solution was provided by ftong.

39. For positive integers n, let $\nu_3(n)$ denote the largest integer k such that 3^k divides n. Find the number of subsets S (possibly containing 0 or 1 elements) of $\{1, 2, \ldots, 81\}$ such that for any distinct $a, b \in S$ with $a > b$, $\nu_3(a-b)$ is even.

Clarifications:

- We only need $\nu_3(a-b)$ to be even for $a > b$.

Solution. The answer is $\boxed{6859000}$.

In general, for $n \geq 0$, we let s_n denote the number of subsets of $\{1, 2, \ldots, 3^n\}$ such that $2 \mid \nu_3(a-b)$ for any distinct $a, b \in S$ and t_n the number of subsets with $2 \nmid \nu_3(a-b)$ for any distinct $a, b \in S$. Clearly $s_0 = t_0 = 2$, and for $n \geq 2$, $s_n = t_{n-1}^3$ while $t_n = 3s_{n-1} - 2$. (Why?) Thus

$$s_4 = t_3^3 = (3s_2 - 2)^3 = (3(3s_0 - 2)^3 - 2)^3 = 190^3 = 6859000$$

is the desired answer.

This problem was proposed by Alex Zhu.

40. Suppose x, y, z, and w are positive reals such that

$$x^2 + y^2 - \frac{xy}{2} = w^2 + z^2 + \frac{wz}{2} = 36$$

and

$$xz + yw = 30.$$

Find the largest possible value of $(xy + wz)^2$.

Solution. The answer is $\boxed{960}$.

Consider a quadrilateral $ABCD$, with $AB = w$, $BC = x$, $CD = y$, $DA = z$, and $AC = 6$. By the law of cosines, the first two equations imply that $\cos B = \frac{1}{4}$ and $\cos D = -\frac{1}{4}$, so $ABCD$ must be cyclic. By Ptolemy's theorem, $BD \cdot AC = wy + xz = 30$, so $BD = 5$.

Note that $\sin B = \sin D = \sqrt{1 - \cos^2 B} = \frac{\sqrt{15}}{4}$, so

$$[ABCD] = [ABC] + [CDA] = \frac{wx \sin B}{2} + \frac{yz \sin D}{2} = \frac{\sqrt{15}}{8}(wx + yz),$$

Let θ be the angle formed by diagonals BD and AC. It is well-known that $[ABCD] = \frac{BD \cdot AC}{2} \sin\theta \leq \frac{BD \cdot AC}{2} = 15$, with equality holding exactly when $\theta = 90^\circ$. It follows that

$$(wx + yz)^2 \leq \frac{64}{15} \cdot 15^2 = 960.$$

Equality holds when $BD \perp AC$; we can ensure that such a configuration exists by holding AC fixed in the circumcircle of $ABCD$, and varying chord BD of length 5 until it is perpendicular to chord AC.

Proof of said well-known fact: Let BD and AC meet at P. Then

$$\begin{aligned}[ABCD] &= [PAB] + [PBC] + [PCD] + [PDA] \\ &= \frac{PA \cdot PB \cdot \sin\theta}{2} + \frac{PB \cdot PC \cdot \sin\theta}{2} + \frac{PC \cdot PD \cdot \sin\theta}{2} + \frac{PD \cdot PA \cdot \sin\theta}{2} \\ &= \frac{(PA + PC)(PB + PD)}{2}\sin\theta = \frac{BD \cdot AC}{2}\sin\theta.\end{aligned}$$

This problem was proposed by Alex Zhu.

41. Find the remainder when

$$\sum_{i=2}^{63} \frac{i^{2011} - i}{i^2 - 1}.$$

is divided by 2016.

Solution. The answer is $\boxed{1011}$.

Solution 1: It is known for odd n and odd k that $1 + 2 + \cdots + n | 1^k + 2^k + \cdots + n^k$. To see this, note that

$$1^k + 2^k + \cdots + n^k = (1^k + (n-1)^k) + (2^k + (n-2)^k) + \cdots + ((\frac{n-1}{2})^k + (\frac{n+1}{2})^k + n^k,$$

and that each summand is a multiple of n, and that

$$1^k + 2^k + \cdots + n^k = (1^k + n^k) + (2^k + (n-1)^k) + \cdots + ((\frac{n-1}{2})^k + (\frac{n+3}{2})^k) + (\frac{n+1}{2})^k,$$

and that each summand is a multiple of $\frac{n+1}{2}$, so $1^k + 2^k + \cdots + n^k$ must be a multiple of $\frac{n(n+1)}{2}$.

Consequently, $1^k + 2^k + \cdots + 63^k \equiv 0 \pmod{2016}$ for all odd k.

Solution 2: Let $P(i) = \frac{i^{2011} - i}{i^2 - 1}$ after dividing out. Note that $P(1) = 1005$ (l'Hopital's rule makes this easy).

Let S be the sum we want. Note that $P(i)$ is an odd function, so $P(i) + P(63 - i) \equiv 0 \pmod{63}$ and we easily get $P(1) + S \equiv P(63) \equiv 0 \pmod{63}$. Similarly, $P(1) + S \equiv 0 \pmod{64}$.

So $S \equiv -1005 \pmod{2016}$ and the answer is 1011.

This problem was proposed by Alex Zhu.

42. In triangle ABC, $\sin\angle A = \frac{4}{5}$ and $\angle A < 90°$ Let D be a point outside triangle ABC such that $\angle BAD = \angle DAC$ and $\angle BDC = 90°$. Suppose that $AD = 1$ and that $\frac{BD}{CD} = \frac{3}{2}$. If $AB + AC$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ where a, b, c are pairwise relatively prime integers, find $a + b + c$?

Solution. The answer is $\boxed{34}$.

Let $A = (0,0), D = (1,0)$. Then let vectors DB and DC be $[-3a, 3b]$ and $[-2b, -2a]$. Already, we have eliminated three of the conditions. Now we have $B = (1 - 3a, 3b), C = (1 - 2b, -2a)$. We know $\tan\frac{A}{2} = \frac{1}{2}$ by half angle formula, so $\frac{3b}{1-3a} = \frac{1}{2}$, $\frac{-2a}{1-2b} = \frac{-1}{2}$. Now we have a linear system of equations which we solve to get $a = \frac{2}{9}, b = \frac{1}{18}$, so $B = \left(\frac{1}{3}, \frac{1}{6}\right), C = \left(\frac{8}{9}, \frac{-4}{9}\right)$. We can finish using distance formula to get $AB + AC = \frac{\sqrt{5}}{6} + \frac{4\sqrt{5}}{9} = \frac{11\sqrt{5}}{18}$ giving an answer of $11 + 5 + 18 = 34$.

Authors Comment: This problem was actually "intended" to be a bash problem. By that, I mean that the more clever your bash, the easier the solution.

This problem was proposed by Ray Li.

43. An integer x is selected at random between 1 and 2011! inclusive. The probability that $x^x - 1$ is divisible by 2011 can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

Solution. The answer is $\boxed{1197}$.

Let $p = 2011$ (which is prime), and for $k = 1, 2, \ldots, p$, let $S_k = \{x \in [1, 2011!] \mid x \equiv k \pmod p\}$ and $T_k = \{x \in S_k : p \mid x^x - 1\}$. As $|S_1| = \cdots = |S_p| = \frac{1}{p}p!$, the desired probability is $\frac{1}{p}\sum_{k=1}^{p} \frac{|T_k|}{|S_k|}$.

Now we observe that for $x \in S_k$, $p \mid x^x - 1$ iff $k \neq 0$ and $\text{ord}_p(k) \mid x$. But $\text{ord}_p(k) \mid p - 1$ by Fermats little theorem and $\gcd(p, p-1) = 1$, so in view of the Chinese remainder theorem and the fact that $S_k = k + p\{0, 1, \ldots, (p-1)! - 1\}$ (note that $\text{ord}_p(k) \mid (p-1)!$), we simply have $\frac{|T_k|}{|S_k|} = \frac{1}{\text{ord}_p(k)}$, whence

$$\frac{1}{p}\sum_{k=1}^{p} \frac{|T_k|}{|S_k|} = \frac{1}{p}\sum_{k=1}^{p-1} \frac{|T_k|}{|S_k|} = \frac{1}{p}\sum_{k=1}^{p-1} \frac{1}{\text{ord}_p(k)}.$$

Since for every positive divisor d of $p-1$, there are exactly $\phi(d)$ (here ϕ denotes Eulers totient function) residues k with $\text{ord}_p(k) = d$ (why?), our answer is

$$\frac{1}{p}\sum_{d \mid p-1} \frac{\phi(d)}{d} = \frac{1}{2011}\sum_{d \mid 2010} \frac{\phi(d)}{d} = \frac{1}{2011}\left(1 + \left(1 - \frac{1}{2}\right)\right)\left(1 + \left(1 - \frac{1}{3}\right)\right)\left(1 + \left(1 - \frac{1}{5}\right)\right)\left(1 + \left(1 - \frac{1}{67}\right)\right),$$

which evaluates to $\frac{1197}{269474}$. (See if you can full justify the last sum on your own.)

This problem was proposed by Alex Zhu.

44. Given a set of points in space, a *jump* consists of taking two points in the set, P and Q, removing P from the set, and replacing it with the reflection of P over Q. Find the smallest number n such that for any set of n lattice points in 10-dimensional-space, it is possible to perform a finite number of jumps so that some two points coincide.

Solution. The answer is $\boxed{1025}$.

Notice that the parity of a point's coordinates does not change when it undergoes a jump, so if we take the unit 10 dimensional cube (all coordinates at 0 or 1), we have 2^{10} points, and no two will ever coincide because that would imply their parities are the same. Thus the answer is at least 1025.

To show that 1025 works, we use induction on the number of dimensions.

Base Case: Any 3 points in 1-dimensional space can be moved so that some two coincide. *Proof:* Jump an outer point over the middle point, maximum distance decreases.

Inductive step: Consider any $2^{n+1} + 1$ points in $(n+1)$-dimensional space. We're going to show that we can make some $2^n + 1$ of them lie in an n-dimensional "plane," so that we can apply the inductive step.

Let's project the points down onto 1-dimensional space; say, by looking only at the first coordinate. Then any jump in $(n+1)$-space corresponds directly to a jump of the projected points (feet) in 1-space.

Now divide up the points into two sets A and B based on the parity of the first coordinate. We can assume WLOG that both A and B are non-empty, else we can either divide by 2, or shift over and divide by 2, and the sequence of moves remains the same. By pigeonhole, we know that one of the sets has at least $2^n + 1$ elements, let that be A. Now, choose any element P of B. By manipulating the feet of any two elements $a_1, a_2 \in A$ and P, we can get the feet of a_1 and a_2 to coincide (by the base case, and parity) and thus a_1 and a_2 lie in the same n-dimensional plane. Now we repeat this algorithm, subsequently always treating a_1 and a_2 as a unit (so that they remain in the same plane whenever they jump over another point).

Thus, by repeatedly manipulating the feet of P and two elements from A, we can accumulate all the points in A into n-space, and we are done. Thus, the answer is 1025.

This problem was proposed by Anderson Wang. The solution was provided by ftong.

45. Let K_1, K_2, K_3, K_4, K_5 be 5 distinguishable keys, and let D_1, D_2, D_3, D_4, D_5 be 5 distinguishable doors. For $1 \leq i \leq 5$, key K_i opens doors D_i and D_{i+1} (where $D_6 = D_1$) and can only be used once. The keys and doors are placed in some order along a hallway. Key$ha walks into the hallway, picks a key and

opens a door with it, such that she never obtains a key before all the doors in front of it are unlocked. In how many such ways can the keys and doors be ordered if Key$ha can open all the doors?

Clarifications:

- The doors and keys are in series. In other words, the doors aren't lined up along the side of the hallway. They are blocking Key$has path to the end, and the only way she can get past them is by getting the appropriate keys along the hallway.
- The doors and keys appear consecutively along the hallway. For example, she might find $K_1D_1K_2D_2K_3D_3K_4D_4K_5D_5$ down the hallway in that order. Also, by "she never obtains a key before all the doors in front of it are unlocked, we mean that she cannot obtain a key before all the doors appearing before the key are unlocked. In essence, it merely states that locked doors cannot be passed.
- The doors and keys do not need to alternate down the hallway.

Solution. The answer is $\boxed{187120}$.

If K_1 is used to unlock D_1, then D_i must be unlocked by K_i for all i; otherwise, if K_1 is used to unlock D_2, then D_{i+1} must be unlocked by K_i for all i. Let A denote the set of proper configurations of the first kind (i.e. in which D_i always appears after K_i), and B of the second (i.e. in which D_{i+1} always appears after K_i); then our answer is $|A \cup B| = |A| + |B| - |A \cap B|$.

First, it's clear that $|A| = |B| = \binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 5!(2 \cdot 5 - 1)!! = 113400$, since proper configurations in A (and B) simply correspond to pairings (K_i, D_i) (resp. (K_i, D_{i+1})) of $\{1, 2, \ldots, 10\}$.

It remains to compute $|A \cap B|$, the number of proper configurations in which D_i, D_{i+1} appear after K_i for all i. It suffices to count the number X of such configurations with D_1 as the first door and D_2 or D_3 as the second, since $5 \cdot 2 \cdot X = |A \cap B|$ by symmetry. Indeed, we can first cyclically shift the indices so that D_1 is the first door, and then observe that the properness-preserving involution taking $K_i \to K_{6-i}$ and $D_i \to D_{7-i}$ (all indices modulo 5) exchanges the second door between the sets $\{D_2, D_3\}$ and $\{D_4, D_5\}$.

We now do casework based on the 5-tuple $(i_1, \ldots, i_5)$ such that $D_{i_1}, \ldots, D_{i_5}$ are in order from left to right. For $(1,2,3,4,5)$ and $(1,2,3,5,4)$, we choose (the positions of) K_1, K_5, K_2, K_3, K_4 (in that order) for a total of $1 \cdot 2 \cdot 4 \cdot 6 \cdot 8 = 384$ configurations; for $(1,2,4,5,3)$ and $(1,2,4,3,5)$, we choose K_1, K_5, K_2, K_3, K_4 for $1 \cdot 2 \cdot 4 \cdot 6 \cdot 7 = 336$; for $(1,2,5,3,4)$ and $(1,2,5,4,3)$, we choose K_1, K_5, K_2, K_4, K_3 for $1 \cdot 2 \cdot 4 \cdot 6 \cdot 8 = 384$; for $(1,3,2,4,5)$ and $(1,3,2,5,4)$, we choose K_1, K_5, K_2, K_3, K_4 for $1 \cdot 2 \cdot 4 \cdot 5 \cdot 8 = 320$; for the remaining four cases (with $i_1 = 1$, $i_2 = 3$, and $i_3 \in \{4, 5\}$), we choose K_1, K_5, K_2, K_3, K_4 for $1 \cdot 2 \cdot 4 \cdot 5 \cdot 7 = 280$.

Thus $X = 384 \cdot 2 + 336 \cdot 2 + 384 \cdot 2 + 320 \cdot 2 + 280 \cdot 4 = 3968$ ways to do this, so $|A \cap B| = 10X = 39680$ and $|A \cup B| = 2 \cdot 113400 - 39680 = 187120$.

Comment: When we replace 5 by n in general, the number $|A \cap B|$ (39680 for $n = 5$) is simply $\frac{1}{2}(2n)E_{2n-1}$, where the Euler number E_n denotes the number of alternating permutations of $\{1, 2, \ldots, n\}$. There is a simple way to recursively compute these numbers. (Try it!)

This problem was proposed by Mitchell Lee. The solution was provided by antimonyarsenide.

46. f is a function from the set of positive integers to itself such that $f(x) \le x^2$ for all natural numbers x, and $f(f(f(x))f(f(y))) = xy$ for all natural numbers x and y. Find the number of possible values of $f(30)$.

Solution. The answer is $\boxed{24}$.

Function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ has the following properties for all x, y:

- $f(x) \le x^2$
- $f(f^2(x)f^2(y)) = xy$

where $f^n(x) = \overbrace{f(f(\cdots f}^{n\ fs}(x)\cdots))$. (1) means that $f(1) = 1$. Plugging $y = 1$ into (2) gives us $x = f(f^2(x)f^2(1)) = f^3(x)$ for all x.

Lemma 1. f is a bijection.

Proof. If $x \neq y$ and $f(ax) = f(y)$, then $x = f^3(x) = f^3(y) = y$, a contradiction, so f is injective. Because $f^3(x) = x$ for all x, then for every x there exists $y = f^2(x)$ such that $f(y) = x$, so f is surjective.

This means we can talk about the inverse function, f^{-1}. Note that $f^{-1} = f^2$.

Lemma 2. $f(x)f(y) = f(xy)$ and $f^{-1}(x)f^{-1}(y) = f^{-1}(xy)$ for all x, y.

Proof. By (2), $f(f^{-1}(x)f^{-1}(y)) = f(f^2(x)f^2(y)) = xy$. Apply f^{-1} to both sides to get $f^{-1}(x)f^{-1}(y) = f^{-1}(xy)$. This also holds true for f itself: $f(x)f(y) = f(f^{-1}(f(x)f(y))) = f(f^{-1}(f(x))f^{-1}(f(y))) = f(xy)$.

In particular, if we know $f(p)$ for all primes p then we know $f(x)$ for all x.

Lemma 3. If p is prime, then $f(p)$ is prime.

Proof. Suppose that $f(p) = qr$ for some prime p and some product qr with $q, r \geq 2$. $p = f^{-1}(qr) = f^{-1}(q)f^{-1}(r)$, so p is the product of two numbers greater than 1 (because $f(1) = 1$), a contradiction.

We can describe all possible f as a set of ordered triples of primes (p, q, r), defining $f(p) = q$, $f(q) = r$ and $f(r) = p$ (and saying WLOG $p < q, r$) such that every prime appears in at most one triple and for all triples, $p^4 \geq q^2 \geq r$. If a prime s appears in no triple, then $f(s) = s$.

Lemma 4. All such sets of triples describe a valid f.

Proof. Consider any $x = \prod p_i$ where each p_i is prime. $f(x) = f\left(\prod p_i\right) = \prod f(p_i) \leq \prod p_i^2 = x^2$, so f satisfies (1). Let $y = \prod q_i$ where each q_i is prime. $f(x)f(y) = \left(\prod f(p_i)\right)\left(\prod f(q_i)\right) = f(xy)$ (because the p_i and q_i together are the prime factors of xy), so f satisfies (2).

We can now consider $f(30) = f(2)f(3)f(5)$. We know $f(2) \in \{2, 3\}$. If $f(2) = 3$, then $f(3) \in \{5, 7\}$. If $f(3) = 5$ then $f(5) = 2$ and $f(30) = 30$; if $f(3) = 7$ then $f(5) \in \{5, 11, 13, 17, 19, 23\}$, each of which gives a new value for $f(30)$. That's 7 values so far, all of them odd except for 30.

If $f(2) = 2$, then $f(3) \in \{3, 5, 7\}$. Either way, $f(5) \in \{5, 7, 11, 13, 17, 19, 23\}$, except

- $f(5) \neq 5$ if $f(3) = 5$.
- $f(5) \neq 7$ if $f(3) = 7$.
- If $f(3) = 3$ and $f(5) = 5$ then $f(30) = 30$, which we've already counted.
- We count $f(30) = 2 \cdot 5 \cdot 7$ twice.

That's a total of $7 \cdot 3 - 4 = 17$ more values, all of them even and none of them 30.

This makes $7 + 17 = 24$ possible values.

This problem was proposed by Alex Zhu. The solution was provided by ziv.

47. Let $ABCD$ be an isosceles trapezoid with bases $AB = 5$ and $CD = 7$ and legs $BC = AD = 2\sqrt{10}$. A circle ω with center O passes through A, B, C, and D. Let M be the midpoint of segment CD, and ray AM meet ω again at E. Let N be the midpoint of BE and P be the intersection of BE with CD. Let Q be the intersection of ray ON with ray DC. There is a point R on the circumcircle of PNQ such that $\angle PRC = 45°$. The length of DR can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. What is $m + n$?

Solution. The answer is $\boxed{122}$.

Since $\triangle BDC$ is oppositely oriented to $\triangle ADC$, we have that BE is a symmedian of $\triangle BDC$. Therefore, $BDEC$ is a harmonic quadrilateral, and so the tangent at B and E to w, and ON meet at Q. This implies BE is the polar of Q with respect to w.

Therefore, $(DPCQ)$ is harmonic, and since CR is an angle bisector of $\angle PRQ$, we have $\angle DRC = 90$. This means that RP is an angle bisector of $\angle DRC$.

We therefore have $\frac{DR}{RC} = \frac{DP}{PC} = \frac{DB^2}{BC^2}$ (by Steiner's Theorem). By Ptolemy's, $DB^2 = (AB)(CD) + (BC)(AD) = 75$. Therefore, $\frac{DR}{RC} = \frac{15}{8}$. Using this along with $DR^2 + RC^2 = 21^2$ (since $\angle DRC = 90$), we have $DR = \frac{105}{17}$, and the answer is 122.

This problem was proposed by Ray Li. The solution was provided by Jinduckey.

48. Suppose that

$$\sum_{i=1}^{982} 7^{i^2}$$

can be expressed in the form $983q + r$, where q and r are integers and $0 \le r \le 492$. Find r.

Solution 1. The answer is $\boxed{450}$.

We start off by brute forcing the facts that $7^{491} \equiv 1 \pmod{983}$, 983 is prime, and 491 is prime.

Now

$$S = \sum_{i=1}^{982} 7^{i^2} = 2\sum_{i=1}^{491} 7^{i^2} = 2\left(1 + 2\sum_{i=1}^{245} 7^{i^2}\right).$$

Consider the set S of all nonzero residues x such that there exists an integer a with $x \equiv a^2 \pmod{491}$. We call S the set of quadratic residues. Let

$$T = \sum_{i=1}^{245} 7^{i^2} = \sum_{k\in S} 7^k.$$

Now, a paper by Monico and Elia shows that any quadratic residue can be expressed as the sum of two other quadratic residues in exactly $d_p - 1 = 122$ ways, and any nonzero $x \notin S$ can be expressed as the sum of two other elements of S in exactly $d_p = 123$ ways. (Try proving this yourself, without resorting to the paper: it's not that difficult!) Since $491 \equiv 3 \pmod 4$, no two quadratic residues sum to zero. Therefore, when we square T, we get

$$T^2 = \sum_{i,j\in S} 7^{i+j} = 122\sum_{k\in S} 7^k + 123\sum_{k\notin S} 7^k = 123\sum_{k=1}^{490} 7^k - \sum_{k\in S} 7^k.$$

Evaluating the geometric series on the right hand side, we get $T^2+T+123 \equiv 0 \pmod{983}$. Multiplying by 4, we have $4T^2 + 4T + 492 \equiv (2T+1)^2 + 491 \equiv 0$, so $(2T+1)^2 \equiv 492 \equiv (S/2)^2$. Then $S \equiv \sqrt{2}$. We find that $S = 450$ works (by raising 2 to the power of 246).

Solution 2. We have $983 = p = 2q + 1$, where $q = 491 \equiv 3 \pmod 4$ and p are prime; since $\left(\frac{7}{p}\right) = 1$, and $p \nmid 7^2 - 1$ (yet $p \mid 7^{2q} - 1$), we have $\operatorname{ord}_p(7) = q$. Of course, it suffices to find $f(7)$ (which is $S/2$ after some simple manipulation, where S is the desired sum), where

$$f(x) = \sum_{k=1}^{q-1}\left(\frac{k}{q}\right)x^k.$$

Clearly $f(1) = 0$ (there are just as many quadratic residues as quadratic nonresidues) and $f(7^r) = \left(\frac{r}{q}\right)f(7)$ if $q \nmid r$. Let $g(x) = x^q - 1 \equiv (x-1)(x-7)\cdots(x-7^{q-1}) \pmod p$, so

$$\prod_{j\neq k}(7^k - 7^j) \equiv g'(7^k) = q7^{k(q-1)} \equiv q7^{-k} \pmod p.$$

Then by Lagrange interpolation over the roots of $g(x)$ (i.e. $7^0, 7^1, \ldots, 7^{q-1}$), we have

$$f(x) \equiv_p f(1)\prod_{j\neq 0}\frac{x-7^j}{1-7^j} + \sum_{k=1}^{q-1} f(7^k)\prod_{j\neq k}\frac{x-7^j}{7^k-7^j} \equiv_p \sum_{k=1}^{q-1}\left(\frac{k}{q}\right)\frac{7^k f(7)}{q}\prod_{j\neq k}(x-7^j).$$

Comparing leading coefficients (i.e. $[x^{490}]$), we find

$$-1 = \left(\frac{q-1}{q}\right) \equiv \frac{f(7)^2}{q} \pmod p;$$

the rest is computation.

Comment: This problem is simply a quadratic Gauss sum in disguise!

This problem was proposed by Alex Zhu. The first solution was provided by ftong. The second solution was provided by math154.

49. Find the magnitude of the product of all complex numbers c such that the recurrence defined by $x_1 = 1$, $x_2 = c^2 - 4c + 7$, and $x_{n+1} = (c^2 - 2c)^2 x_n x_{n-1} + 2x_n - x_{n-1}$ also satisfies $x_{1006} = 2011$.

Solution. The answer is $\boxed{2010}$.

First we observe that for every $n \geq 1$, there exists a monic polynomial $P_n \in \mathbb{Z}[x]$ such that $x_n = P_n(c)$ for all complex numbers c. By a simple induction, $P_n(0) = 6n - 5$ for $n \geq 1$, so $P_{1006}(0) = 6031$. Similarly, we can show that $(x-2)^2 \mid P_n(x) - 2n + 1$ for $n \geq 1$: just note that $(x-2)^2 \mid P_{n+1}(x) - 2P_n(x) + P_{n-1}(x)$ by the recursion.

Write $P_{1006}(x) - 2011$ (which is monic with integer coefficients) in the form $\prod_{i=1}^m f_i(x)^{k_i}$ for distinct nonconstant monic irreducible polynomials $f_i \in \mathbb{Z}[x]$ and positive integers k_i; the desired product of c such that $P_{1006}(c) = 2011$ is simply $P = \prod_{i=1}^m |f_i(0)|$, since no monic irreducible polynomial has a double root, and no two distinct monic irreducible polynomials with integer coefficients can share a root. (Prove this!)

Note that for any i, $f_i(0)^{k_i} \mid P_{1006}(0) - 2011 = 4020 = 2^2 \cdot 3 \cdot 5 \cdot 67$. If $k_i > 1$ for some i, then we must in fact have $f_i(0) \mid 2$ (and if $f_i(0) = \pm 2$, then we must also have $k_i = 2$). Thus for every i, $|f_i(0)^{k_i}| = |f_i(0)|$ unless $f_i(0) = \pm 2$ and $k_i = 2$ (in which case $|f_i(0)^{k_i}| = 2|f_i(0)|$), so $P = \prod_{i=1}^m |f_i(0)| = \frac{1}{2^r} \prod_{i=1}^m |f_i(x)^{k_i}| = \frac{4020}{2^r}$, where r is the number of i such that $f_i(0) = \pm 2$ and $k_i = 2$. Since $(2^2)^2 \nmid 4020$, we have $r \leq 1$. But we know $(x-2)^2 \mid P_{1006}(x) - 2011$ from above, so $r \geq 1 \implies r = 1$, and $P = 2010$.

This problem was proposed by Alex Zhu.

50. In tetrahedron $SABC$, the circumcircles of faces SAB, SBC, and SCA each have radius 108. The inscribed sphere of $SABC$, centered at I, has radius 35. Additionally, $SI = 125$. Let R is the largest possible value of the circumradius of face ABC. Give that R can be expressed in the form $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers, find $m + n$.

Solution. The answer is $\boxed{35928845209}$.

The answer is $\sqrt{\frac{35925411600}{3433609}}$.

Let $r = 35$, $s = 125$, $a = 108$, $v = \sqrt{s^2 - r^2} = 120$, ω be the insphere of $SABC$, and u be the circumradius of $SABC$. Let O be the circumcenter of $SABC$, and let O_A, O_B, and O_C be the circumcenters of faces SBC, SCA, and SAB, respectively. Since OO_A is perpendicular to face SBC, by the Pythagorean theorem $OO_A^2 = u^2 - a^2$. Similarly, we must have $OO_B^2 = OO_C^2 = u^2 - a^2$, so O is equidistant from the three faces of $SABC$. Thus, S, I, and O are collinear. If we let the tangency point of the insphere with $SABC$ be I_A, then $\triangle SI_AI \sim \triangle SO_AO$, so $\frac{s}{u} = \frac{SI}{SO} = \frac{r}{\sqrt{u^2 - a^2}}$, whence $\frac{u^2}{u^2 - a^2} = \frac{s^2}{r^2}$. Solving, we get $u = \frac{as}{v} = \frac{225}{2}$.

Let h be the altitude from S to face ABC. We claim that the altitude of h is uniquely determined from the information given. To see this, we perform an inversion centered about S with radius 1. For any point or solid X in the original diagram, let X' denote its inverse under this inversion. Let Γ be the circumsphere of $SA'B'C'$, and let K be its center and R' be its circumradius. Γ is the image of plane ABC under the inversion. ω' is externally tangent to Γ at a point T, and tangent to faces $SA'B'$,$SB'C'$, and $SC'B'$. We first compute the radius of ω' and the distance SI'.

Let line SI meet ω at A and B, with A closer to S than B. We have $SA = s - r$ and $SB = s + r$. Thus, $SA' = \frac{1}{s-r}$ and $SB' = \frac{1}{s+r}$. Consequently, the radius r' of ω' is $\frac{\frac{1}{s-r} - \frac{1}{s+r}}{2} = \frac{r}{v^2}$, and the distance s' from S to I' is $\frac{\frac{1}{s-r} + \frac{1}{s+r}}{2} = \frac{s}{v^2}$.

Let F be the foot of the perpendicular from S to face $A'B'C'$. F is the image of the antipode of S in the circumsphere of $SABC$, so $SF = \frac{1}{2u}$. Since S, O, and I are collinear, S, F, and I' are collinear. Thus, SI' is perpendicular to face $A'B'C'$.

Draw all the lines through S tangent to ω. Because SI' is perpendicular to plane $A'B'C'$, the intersections of these lines with plane $A'B'C'$ will be a circle ω_0. Let any of these tangent lines meet ω' at P and $A'B'C'$ at Q. ω' is the incircle of $\triangle A'B'C'$. Also, FQ is a radius (since F is the center of ω_0, as S, F, and I' are collinear.) We have $\triangle SQF \sim \triangle SI'P$, so $\frac{FQ}{FS} = \frac{I'P}{SP}$. Thus,

$$FQ = FS \cdot \frac{r'}{\sqrt{s'^2 - r'^2}} = \frac{1}{2u} \cdot \frac{r/v^2}{\sqrt{(s^2-r^2)/v^4}} = \frac{r}{2uv}.$$

Let O_0 be the center of the circumcircle of $A'B'C'$, and let x be its radius. By Euler's distance formula applied to $A'B'C'$, we have that $O_0F^2 = x(x - \frac{r}{uv})$. Now, consider the cross-section of Γ on the plane through S and K perpendicular to plane $A'B'C'$, which must thus pass through O_0, F, and I'. In this cross-section there is a circle with radius R' centered at K'. AB is a chord inside the circle, O_0 is its midpoint, $AO_0 = x$, $O_0F = \sqrt{x(x - \frac{r}{uv})}$, and S is a point on the circle on the same side of AB as K with distance $\frac{1}{2u}$ from AB. There is a circle centered at I' with radius r' tangent to the circle, and $SI' = s'$. Let J be the foot of the perpendicular from K to SJ.

By the Pythagorean theorem,

$$R'^2 = KS^2 = SJ^2 + KJ^2 = (SF - JF)^2 + O_0F^2 = (\frac{1}{2u} - \sqrt{R'^2 - x^2})^2 + x(x - \frac{r}{uv}),$$

and

$$(R' + r')^2 = KI'^2 = JI'^2 + KJ^2 = (SI' - SJ)^2 + x(x - \frac{r}{uv}).$$

We leave the relatively straightforward but tedious computations to the interested reader.

This problem was proposed by Alex Zhu.