The Online Math Open
January 16-23, 2012

Online Math Open

Contest Information

Format

The test will start Monday January 16 and end Monday January 23. The test consists of 50 short answer questions, each of which has a nonnegative integer answer. The problem difficulties range from those of AMC problems to those of Olympiad problems. Problems are ordered in roughly increasing order of difficulty.

Team Guidelines

Students may compete in teams of up to four people. Participating students must not have graduated from high school. International students may participate. No student can be a part of more than one team. The members of each team do not get individual accounts; they will all share the team account.

The team will submit their final answers through their account. Though teams can save drafts for their answers, the current interface does not allow for much flexibility in communication between team members. We recommend using Google Docs and Spreadsheets to discuss problems and compare answers, especially if teammates cannot communicate in person. Teams may spend as much time as they like on the test before the deadline.

Aids

Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids are not allowed. This is includes (but is not limited to) Geogebra and graphing calculators. Published print and electronic resources are permitted.

Four-function calculators are permitted on the Online Math Open. That is, calculators which perform only the four basic arithmetic operations (+-*/) may be used. No other computational aids such as scientific and graphing calculators, computer programs and applications such as Mathematica, and online databases are permitted. All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.

Scoring

Each problem will be worth one point. Ties will be broken based on the highest problem number that a team answered correctly. If there are still ties, those will be broken by the second highest problem solved, and so on.

Results

After the contest is over, we will release the answers to the problems within the next day. If you have a protest about an answer, you may send an email to onlinemathopen2011@gmail.com (Include "Protest" in the subject). Solutions and results will be released in the following weeks.

1. The average of two positive real numbers is equal to their difference. What is the ratio of the larger number to the smaller one?

2. How many ways are there to arrange the letters A, A, A, H, H in a row so that the sequence HA appears at least once?

3. A lucky number is a number whose digits are only 4 or 7. What is the 17th smallest lucky number?

4. How many positive even numbers have an even number of digits and are less than 10000?

5. Congruent circles Γ_1 and Γ_2 have radius 2012, and the center of Γ_1 lies on Γ_2. Suppose that Γ_1 and Γ_2 intersect at A and B. The line through A perpendicular to AB meets Γ_1 and Γ_2 again at C and D, respectively. Find the length of CD.

6. Alice's favorite number has the following properties:

   - It has 8 distinct digits.
   - The digits are decreasing when read from left to right.
   - It is divisible by 180.

   What is Alice's favorite number?

7. A board 64 inches long and 4 inches high is inclined so that the long side of the board makes a 30 degree angle with the ground. The distance from the highest point on the board to the ground can be expressed in the form $a + b\sqrt{c}$ where a, b, c are positive integers and c is not divisible by the square of any prime. What is $a + b + c$?

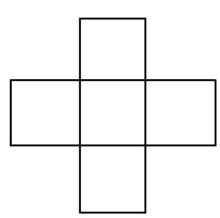
8. An $8 \times 8 \times 8$ cube is painted red on 3 faces and blue on 3 faces such that no corner is surrounded by three faces of the same color. The cube is then cut into 512 unit cubes. How many of these cubes contain both red and blue paint on at least one of their faces?

9. At a certain grocery store, cookies may be bought in boxes of 10 or 21. What is the minimum positive number of cookies that must be bought so that the cookies may be split evenly among 13 people?

10. A drawer has 5 pairs of socks. Three socks are chosen at random. If the probability that there is a pair among the three is $\frac{m}{n}$, where m and n are relatively prime positive integers, what is $m + n$?

11. If
$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{4x^3} + \frac{1}{8x^4} + \frac{1}{16x^5} + \cdots = \frac{1}{64},$$
and x can be expressed in the form $\frac{m}{n}$, where m, n are relatively prime positive integers, find $m + n$.

12. A *cross-pentomino* is a shape that consists of a unit square and four other unit squares each sharing a different edge with the first square. If a cross-pentomino is inscribed in a circle of radius R, what is $100R^2$?



13. A circle ω has center O and radius r. A chord BC of ω also has length r, and the tangents to ω at B and C meet at A. Ray AO meets ω at D past O, and ray OA meets the circle centered at A with radius AB at E past A. Compute the degree measure of $\angle DBE$.

14. Al told Bob that he was thinking of 2011 distinct positive integers. He also told Bob the sum of those integers. From this information, Bob was able to determine all 2011 integers. How many possible sums could Al have told Bob?

15. Five bricklayers working together finish a job in 3 hours. Working alone, each bricklayer takes at most 36 hours to finish the job. What is the smallest number of minutes it could take the fastest bricklayer to complete the job alone?

16. Let $A_1B_1C_1D_1A_2B_2C_2D_2$ be a unit cube, with $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ opposite square faces, and let M be the center of face $A_2B_2C_2D_2$. Rectangular pyramid $MA_1B_1C_1D_1$ is cut out of the cube. If the surface area of the remaining solid can be expressed in the form $a+\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime, find $a+b$.

17. Each pair of vertices of a regular 10-sided polygon is connected by a line segment. How many unordered pairs of distinct parallel line segments can be chosen from these segments?

18. The sum of the squares of three positive numbers is 160. One of the numbers is equal to the sum of the other two. The difference between the smaller two numbers is 4. What is the difference between the cubes of the smaller two numbers?

19. There are 20 geese numbered 1 through 20 standing in a line. The even numbered geese are standing at the front in the order $2, 4, \ldots, 20$, where 2 is at the front of the line. Then the odd numbered geese are standing behind them in the order, $1, 3, 5, \ldots, 19$, where 19 is at the end of the line. The geese want to rearrange themselves in order, so that they are ordered $1, 2, \ldots, 20$ (1 is at the front), and they do this by successively swapping two adjacent geese. What is the minimum number of swaps required to achieve this formation?

20. Let ABC be a right triangle with a right angle at C. Two lines, one parallel to AC and the other parallel to BC, intersect on the hypotenuse AB. The lines cut the triangle into two triangles and a rectangle. The two triangles have areas 512 and 32. What is the area of the rectangle?

21. If

$$2011^{2011^{2012}} = x^x$$

for some positive integer x, how many positive integer factors does x have?

22. Find the largest prime number p such that when $2012!$ is written in base p, it has at least p trailing zeroes.

23. Let ABC be an equilateral triangle with side length 1. This triangle is rotated by some angle about its center to form triangle DEF. The intersection of ABC and DEF is an equilateral hexagon with an area that is $\frac{4}{5}$ the area of ABC. The side length of this hexagon can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. What is $m+n$?

24. Find the number of ordered pairs of positive integers (a, b) with $a+b$ prime, $1 \le a, b \le 100$, and $\frac{ab+1}{a+b}$ is an integer.

25. Let a, b, c be the roots of the cubic $x^3 + 3x^2 + 5x + 7$. Given that P is a cubic polynomial such that $P(a) = b + c$, $P(b) = c + a$, $P(c) = a + b$, and $P(a + b + c) = -16$, find $P(0)$.

26. Xavier takes a permutation of the numbers 1 through 2011 at random, where each permutation has an equal probability of being selected. He then cuts the permutation into increasing contiguous subsequences, such that each subsequence is as long as possible. Compute the expected number of such subsequences.

27. Let a and b be real numbers that satisfy

$$a^4 + a^2b^2 + b^4 = 900,$$
$$a^2 + ab + b^2 = 45.$$

Find the value of $2ab$.

28. A fly is being chased by three spiders on the edges of a regular octahedron. The fly has a speed of 50 meters per second, while each of the spiders has a speed of r meters per second. The spiders choose the (distinct) starting positions of all the bugs, with the requirement that the fly must begin at a vertex. Each bug knows the position of each other bug at all times, and the goal of the spiders is for at least one of them to catch the fly. What is the maximum c so that for any $r < c$, the fly can always avoid being caught?

29. How many positive integers a with $a \le 154$ are there such that the coefficient of x^a in the expansion of

$$(1 + x^7 + x^{14} + \cdots + x^{77})(1 + x^{11} + x^{22} + \cdots + x^{77})$$

is zero?

30. The Lattice Point Jumping Frog jumps between lattice points in a coordinate plane that are exactly 1 unit apart. The Lattice Point Jumping Frog starts at the origin and makes 8 jumps, ending at the origin. Additionally, it never lands on a point other than the origin more than once. How many possible paths could the frog have taken?

31. Let ABC be a triangle inscribed in circle Γ, centered at O with radius 333. Let M be the midpoint of AB, N be the midpoint of AC, and D be the point where line AO intersects BC. Given that lines MN and BO concur on Γ and that $BC = 665$, find the length of segment AD.

32. The sequence $\{a_n\}$ satisfies $a_0 = 201, a_1 = 2011$, and $a_n = 2a_{n-1} + a_{n-2}$ for all $n \ge 2$. Let

$$S = \sum_{i=1}^{\infty} \frac{a_{i-1}}{a_i^2 - a_{i-1}^2}$$

What is $\frac{1}{S}$?

33. You are playing a game in which you have 3 envelopes, each containing a uniformly random amount of money between 0 and 1000 dollars. (That is, for any real $0 \le a < b \le 1000$, the probability that the amount of money in a given envelope is between a and b is $\frac{b-a}{1000}$.) At any step, you take an envelope and look at its contents. You may choose either to keep the envelope, at which point you finish, or discard it and repeat the process with one less envelope. If you play to optimize your expected winnings, your expected winnings will be E. What is $\lfloor E \rfloor$, the greatest integer less than or equal to E?

34. Let p, q, r be real numbers satisfying

$$\frac{(p+q)(q+r)(r+p)}{pqr} = 24$$
$$\frac{(p-2q)(q-2r)(r-2p)}{pqr} = 10.$$

Given that $\frac{p}{q} + \frac{q}{r} + \frac{r}{p}$ can be expressed in the form $\frac{m}{n}$, where m, n are relatively prime positive integers, compute $m + n$.

35. Let $s(n)$ be the number of 1's in the binary representation of n. Find the number of ordered pairs of integers (a, b) with $0 \leq a < 64, 0 \leq b < 64$ and $s(a+b) = s(a) + s(b) - 1$.

36. Let s_n be the number of solutions to $a_1 + a_2 + a_3 + a_4 + b_1 + b_2 = n$, where a_1, a_2, a_3 and a_4 are elements of the set $\{2, 3, 5, 7\}$ and b_1 and b_2 are elements of the set $\{1, 2, 3, 4\}$. Find the number of n for which s_n is odd.

37. In triangle ABC, $AB = 1$ and $AC = 2$. Suppose there exists a point P in the interior of triangle ABC such that $\angle PBC = 70°$, and that there are points E and D on segments AB and AC, such that $\angle BPE = \angle EPA = 75°$ and $\angle APD = \angle DPC = 60°$. Let BD meet CE at Q, and let AQ meet BC at F. If M is the midpoint of BC, compute the degree measure of $\angle MPF$.

38. Let S denote the sum of the 2011th powers of the roots of the polynomial $(x - 2^0)(x - 2^1) \cdots (x - 2^{2010}) - 1$. How many 1's are in the binary expansion of S?

39. For positive integers n, let $\nu_3(n)$ denote the largest integer k such that 3^k divides n. Find the number of subsets S (possibly containing 0 or 1 elements) of $\{1, 2, \ldots, 81\}$ such that for any distinct $a, b \in S$, $\nu_3(a - b)$ is even.

40. Suppose x, y, z, and w are positive reals such that

$$x^2 + y^2 - \frac{xy}{2} = w^2 + z^2 + \frac{wz}{2} = 36$$
$$xz + yw = 30.$$

Find the largest possible value of $(xy + wz)^2$.

41. Find the remainder when

$$\sum_{i=2}^{63} \frac{i^{2011} - i}{i^2 - 1}.$$

is divided by 2016.

42. In triangle ABC, $\sin \angle A = \frac{4}{5}$ and $\angle A < 90°$ Let D be a point outside triangle ABC such that $\angle BAD = \angle DAC$ and $\angle BDC = 90°$. Suppose that $AD = 1$ and that $\frac{BD}{CD} = \frac{3}{2}$. If $AB + AC$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ where a, b, c are pairwise relatively prime integers, find $a + b + c$?

43. An integer x is selected at random between 1 and 2011! inclusive. The probability that $x^x - 1$ is divisible by 2011 can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

44. Given a set of points in space, a *jump* consists of taking two points in the set, P and Q, removing P from the set, and replacing it with the reflection of P over Q. Find the smallest number n such that for any set of n lattice points in 10-dimensional-space, it is possible to perform a finite number of jumps so that some two points coincide.

45. Let K_1, K_2, K_3, K_4, K_5 be 5 distinguishable keys, and let D_1, D_2, D_3, D_4, D_5 be 5 distinguishable doors. For $1 \leq i \leq 5$, key K_i opens doors D_i and D_{i+1} (where $D_6 = D_1$) and can only be used once. The keys and doors are placed in some order along a hallway. Key$ha walks into the hallway, picks a key and opens a door with it, such that she never obtains a key before all the doors in front of it are unlocked. In how many such ways can the keys and doors be ordered if Key$ha can open all the doors?

46. Let f is a function from the set of positive integers to itself such that $f(x) \leq x^2$ for all natural numbers x, and $f\left(f(f(x))f(f(y))\right) = xy$ for all natural numbers x and y. Find the number of possible values of $f(30)$.

47. Let $ABCD$ be an isosceles trapezoid with bases $AB = 5$ and $CD = 7$ and legs $BC = AD = 2\sqrt{10}$. A circle ω with center O passes through A, B, C, and D. Let M be the midpoint of segment CD, and ray AM meet ω again at E. Let N be the midpoint of BE and P be the intersection of BE with CD. Let Q be the intersection of ray ON with ray DC. There is a point R on the circumcircle of PNQ such that $\angle PRC = 45^\circ$. The length of DR can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. What is $m + n$?

48. Suppose that

$$\sum_{i=1}^{982} 7^{i^2}$$

can be expressed in the form $983q + r$, where q and r are integers and $0 \leq r \leq 492$. Find r.

49. Find the magnitude of the product of all complex numbers c such that the recurrence defined by $x_1 = 1$, $x_2 = c^2 - 4c + 7$, and $x_{n+1} = (c^2 - 2c)^2 x_n x_{n-1} + 2x_n - x_{n-1}$ also satisfies $x_{1006} = 2011$.

50. In tetrahedron $SABC$, the circumcircles of faces SAB, SBC, and SCA each have radius 108 The inscribed sphere of $SABC$, centered at I, has radius 35. Additionally, $SI = 125$. Let R is the largest possible value of the circumradius of face ABC. Give that R can be expressed in the form $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers, find $m + n$.

Acknowledgments

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