The Online Math Open Spring Contest
Official Solutions
March 23 – April 3, 2018

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1. Farmer James has three types of cows on his farm. A cow with zero legs is called a *ground beef*, a cow with one leg is called a *steak*, and a cow with two legs is called a *lean beef*. Farmer James counts a total of 20 cows and 18 legs on his farm. How many more *ground beef*s than *lean beef*s does Farmer James have?

   Proposed by James Lin.

   Answer. $\boxed{2}$.

   Solution. Let there be x *ground beef*s, y *steak*s, and z *lean beef*s. Then it follows that $x+y+z=20$ and $y+2z=18$, so subtracting gives $x-z=2$. ☐

2. The area of a circle (in square inches) is numerically larger than its circumference (in inches). What is the smallest possible integral area of the circle, in square inches?

   Proposed by James Lin.

   Answer. $\boxed{13}$.

   Solution. We have that $r^2\pi > 2r\pi \iff r > 2 \iff A > 4\pi \approx 12.57$, so the minimal area is 13. ☐

3. Hen Hao randomly selects two distinct squares on a standard 8×8 chessboard. Given that the two squares touch (at either a vertex or a side), the probability that the two squares are the same color can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Find $100m+n$.

   Proposed by James Lin.

   Answer. $\boxed{715}$.

   Solution. Hen Hao is randomly choosing among pairs of touching squares, so it suffices to count the number of pairs of touching squares of the same color, and of different colors. By considering pairs of rows, there are $7 \times 7 = 49$ pairs of touching black squares, and similarly 49 such pairs of white squares. There are $7 \times 8 = 56$ pairs of horizontally touching squares of opposite colors, and another 56 for vertically touching squares. Hence, our answer is $\frac{2\cdot 49}{2\cdot 49+2\cdot 56} = \frac{7}{15}$, so the answer is $7 \cdot 100 + 15 = 715$. ☐

4. Define $f(x) = |x-1|$. Determine the number of real numbers x such that $f(f(\cdots f(f(x))\cdots)) = 0$, where there are 2018 f's in the equation.

   Proposed by Yannick Yao.

   Answer. $\boxed{2018}$.

   Solution. We work backwards: the root to $f(x) = 0$ is $x = 1$, the roots to $f(f(x)) = 0$ are $x = 0, 2$, the roots to $f(f(f(x))) = 0$ are $x = -1, 1, 3$, etc. We can find a pattern that when there are k applications of f's, there will be k distinct roots that form an arithmetic sequence with median 1 and common difference 2. So there are 2018 roots to the original equation, which are $-2016, -2014, \ldots, 0, 2, \ldots, 2016, 2018$. ☐

5. A mouse has a wheel of cheese which is cut into 2018 slices. The mouse also has a 2019-sided die, with faces labeled $0, 1, 2, \ldots, 2018$, and with each face equally likely to come up. Every second, the mouse rolls the dice. If the dice lands on k, and the mouse has at least k slices of cheese remaining, then the mouse eats k slices of cheese; otherwise, the mouse does nothing. What is the expected number of seconds until all the cheese is gone?

   Proposed by Brandon Wang.

   Answer. $\boxed{2019}$.

Solution. Each second, if there are s slices of cheese remaining, then the mouse must roll exactly s to win that second; thus, the mouse has a $\frac{1}{2019}$ chance of winning on each turn. Then the expected number of seconds $E[X]$ is just

$$\sum_{n=0}^{\infty} \mathbb{P}(X > n) = \sum_{n=0}^{\infty} \left(\frac{2018}{2019}\right)^n = \frac{1}{1 - \frac{2018}{2019}} = 2019.$$

$\square$

6. Let $f(x) = x^2 + x$ for all real x. There exist positive integers m and n, and distinct nonzero real numbers y and z, such that $f(y) = f(z) = m + \sqrt{n}$ and $f(\frac{1}{y}) + f(\frac{1}{z}) = \frac{1}{10}$. Compute $100m + n$.

   Proposed by Luke Robitaille.

   Answer. $\boxed{1735}$.

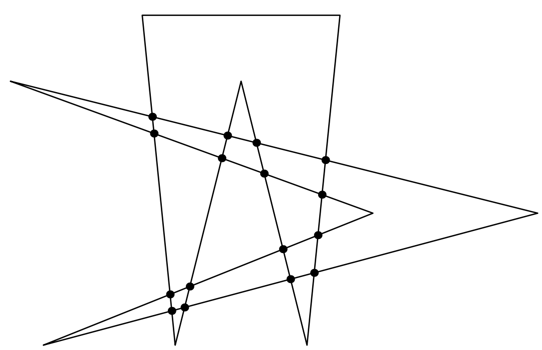
   Solution. Let $X = m + \sqrt{n}$. Since $y \neq z$, $y + z = -1$ and $yz = -X$. Now $\frac{1}{10} = f(\frac{1}{y}) + f(\frac{1}{z}) = \frac{y^2+z^2}{(yz)^2} + \frac{y+z}{yz} = \frac{(y+z)^2}{(yz)^2} - \frac{2}{yz} + \frac{y+z}{yz} = \frac{1}{X^2} + \frac{3}{X}$. Then $X^2 = 30X + 10$, so $X = 15 + \sqrt{235}$, so the answer is $100 * 15 + 235 = 1735$. $\square$

7. A quadrilateral and a pentagon (both not self-intersecting) intersect each other at N distinct points, where N is a positive integer. What is the maximal possible value of N?

   Proposed by James Lin.

   Answer. $\boxed{16}$.

   Solution. Any line can intersect each side of the pentagon at most once. If the line intersects each side of the pentagon at a distinct point (and hence does not pass through a vertex), then this is a contradiction since the line must pass through an even number of sides. (Since each time the line intersects a side, it goes inside the polygon to outside the polygon or vice versa.) Thus, each side of the quadrilateral can intersect the pentagon in at most 4 points, so there are at most 16 intersection points. We can construct the equality case as shown below.



$\square$

8. Compute the number of ordered quadruples (a, b, c, d) of distinct positive integers such that $\binom{\binom{a}{b}}{\binom{c}{d}} = 21$.

   Proposed by Luke Robitaille.

   Answer. $\boxed{13}$.

Solution. The only solutions to $\binom{x}{y} = 21$ are $(x, y) = (21, 1), (21, 20), (7, 2), (7, 5)$. Now we dive into casework based on x.

Case 1: $(x, y) = (21, 1)$ or $(21, 20)$.

Since a, b, c, d are distinct, $y \neq 1$. For $y = 20$, we have the options $(a, b) = (21, 1), (21, 20), (7, 2), (7, 4)$ and $(c, d) = (20, 1), (20, 19), (6, 3)$. This gives a total of $2 + 1 + 3 + 3 = 9$ solutions for (a, b, c, d).

Case 2: $(x, y) = (7, 2)$ or $(7, 5)$.

We have the options $(a, b) = (7, 1), (7, 6)$ and $(c, d) = (2, 1), (5, 1), (5, 4)$, giving 4 solutions.

We have a total of $9 + 4 = 13$ solutions for (a, b, c, d). $\square$

9. Let k be a positive integer. In the coordinate plane, circle ω has positive integer radius and is tangent to both axes. Suppose that ω passes through $(1, 1000 + k)$. Compute the smallest possible value of k.

   Proposed by Luke Robitaille.

   Answer. $\boxed{58}$.

   Solution. Solution 1: Let ω have radius r. Then, it has center (r, r), so that $(r-1)^2+(1000+k-r)^2 = r^2$, i.e., $(1000 + k - r)^2 = 2r - 1$, and so solving for k gives: $k = \pm\sqrt{2r - 1} + r - 1000$. As $2r - 1$ is a perfect square, $r = 2x^2+2x+1$ for some positive integer k, so that $k = \pm(2x+1)+2x^2+2x+1-1000 = 2x^2 - 1000$. (Here we took the plus sign; the minus sign is accounted for by replacing x with $x + 1$.) k is first positive when $x = 23$, which gives $k = 58$.

   Solution 2: Let the circle have radius r and center (r, r). We have $r^2 = (r - 1)^2 + (r - (1000 + k))^2$. This is equivalent to $(r-(1+(1000+k)))^2 = 2*1*(1000+k)$, so if r is an integer, then $2(1000+k)$ is a perfect square. Conversely, if $2(1000+k)$ is a perfect square, then $r = 1+(1000+k)+\sqrt{2(1000 + k)}$ works. Thus the desired condition becomes $2(1000 + k)$ is a perfect square, so the answer is 58. $\square$

10. The one hundred U.S. Senators are standing in a line in alphabetical order. Each senator either always tells the truth or always lies. The ith person in line says:

    "Of the $101 - i$ people who are not ahead of me in line (including myself), more than half of them are truth-tellers."

    How many possibilities are there for the set of truth-tellers on the U.S. Senate?

    Proposed by James Lin.

    Answer. $\boxed{101}$.

    Solution. Let the answer be $f(n)$ when there are $2n$ senators; we want to find the value of $f(50)$. The $2n$th Senator in line may be either a truth-teller or liar. If he/she is a liar, then the rest of the senators must all be liars as well, since the first truth-teller would be a contradiction. If he/she is a truth-teller, then we consider the $2n - 1$th Senator. If the $2n - 1$th Senator is a truth-teller, then the rest of the senators must all be truth-tellers as well since the first liar would be a contradiction. If the he/she is a liar, then we may effectively erase these last two senators, which will not affect the statements of the first $2n - 2$ senators. Thus, $f(n) = f(n - 1) + 2$ for all $n \geq 2$. Since $f(1) = 3$, it follows that $f(50) = 101$. $\square$

11. Lunasa, Merlin, and Lyrica are performing in a concert. Each of them will perform two different solos, and each pair of them will perform a duet, for nine distinct pieces in total. Since the performances are very demanding, no one is allowed to perform in two pieces in a row. In how many different ways can the pieces be arranged in this concert?

    Proposed by Yannick Yao.

    Answer. $\boxed{384}$.

Solution. Notice that if a duet not at the beginning or end of the concert must be preceded and followed by the two solos performed by the other performer not in the duet. We now casework on the number of duets at either end of the concert.

(During the casework, we assume that the two solos of each performer are identical, so we need to multiply the number by $2^3 = 8$ at the end.)

Case 0: No duets are at beginning or end of the concert. Since each of them need to be sandwiched by two solos, forming three blocks of three pieces, there are $3! = 6$ ways to arrange the blocks.

Case 1: One duet is at the beginning or end of the concert. WLOG assume that it's Lunasa and Merlin's duet at the beginning of the concert (there are 6 ways to the placing and performers of this duet). Then this duet must be followed by one of Lyrica's solo, and the other two duets still form blocks of three pieces. The other Lyrica's duet can be placed between these blocks arbitrarily, except immediately after her first solo. There are 4 ways to arrange the blocks, so there are 24 ways in total.

Case 2: One duet is at each of the beginning and the end of concert. There are $3 \times 2 = 6$ ways to choose the two duets. WLOG assume that the beginning is Lunasa and Merlin's duet and the ending is Lunasa and Lyrica's duet. Then the second show must be Lyrica's solo and the penultimate show must be Merlin's solo. Lunasa's two solos must sandwich Merlin and Lyrica's duet, and there are 3 ways to place this block. It is not difficult to see that each placement produce exactly one way to place the remaining two solos, so there are 18 ways in total.

Thus, adding all cases up, we have $8(6 + 24 + 18) = 384$ ways in total. □

12. Near the end of a game of Fish, Celia is playing against a team consisting of Alice and Betsy. Each of the three players holds two cards in their hand, and together they have the Nine, Ten, Jack, Queen, King, and Ace of Spades (this set of cards is known by all three players). Besides the two cards she already has, each of them has no information regarding the other two's hands (In particular, teammates Alice and Betsy do not know each other's cards).

It is currently Celia's turn. On a player's turn, the player must ask a player on the other team whether she has a certain card that is in the set of six cards but *not* in the asker's hand. If the player being asked does indeed have the card, then she must reveal the card and put it in the askeras hand, and the asker shall ask again (but may ask a different player on the other team); otherwise, she refuses and it is now her turn. Moreover, a card may not be asked if it is known (to the asker) to be not in the asked person's hand. The game ends when all six cards belong to one team, and the team with all the cards wins. Under optimal play, the probability that Celia wins the game is $\frac{p}{q}$ for relatively prime positive integers p and q. Find $100p + q$.

Proposed by Yannick Yao.

Answer. $\boxed{1318}$.

Solution. Note that if a card is asked, then regardless of outcome the location of this card is now publicly known. Therefore the game is determined as soon as a person's hand is entirely known, and the person who currently has the turn can win the game since he/she now know the cards in all three person's hands. (Unless the person is the one whose hand is entirely known publicly, which will not happen in our discussion below.)

Celia starts the turn knowing nothing, so by symmetry we may assume WLOG she asks Alice for a card A, which Alice will have with probability $\frac{1}{2}$.

Case 1: Alice does not have the card A, then A must belong to Betsy's hand. It is currently Alice's turn, and she can only ask Celia for cards, if Alice ever asks for a card that Celia does not have, then that card is in Betsy's hand, and since it's now Celia's turn Celia will win the game. Therefore for Alice to win she need to guess Celia's entire hand correctly, which happens with probability $\frac{1}{3}$. So Celia wins with probability $\frac{2}{3}$ in this case.

Case 2: Alice does have the card A, then Celia can ask for another card.

Case 2a: Celia asks Alice for another card. If Celia succeeds (with probability $\frac{1}{3}$), then Celia wins immediately. Otherwise, it is Alice's turn and by Case 1 Celia wins with probability $\frac{2}{3}$. The probability that Celia wins in total is $\frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3} = \frac{7}{9}$.

Case 2b: Celia asks Betsy for another card. Note that in this case Celia wins if and only if she manages to guess Betsy's hand correctly, which happens with probability $\frac{1}{3}$. (If Celia fails with the first question, then Alice's hand is entirely known and Betsy will use that information to win. If Celia fails with the second question, then either Alice's or Betsy's hand will also be entirely known, and their team will win.)

Therefore it is optimal for Celia to ask Alice. (So Case 2b shall be disregarded.)

Since Case 1 and 2a happen with equal probability, the total probability is $\frac{1}{2}(\frac{2}{3} + \frac{7}{9}) = \frac{13}{18}$. So the requested answer is 1318. $\square$

13. Find the smallest positive integer n for which the polynomial

$$x^n - x^{n-1} - x^{n-2} - \cdots - x - 1$$

has a real root greater than 1.999.

Proposed by James Lin.

Answer. $\boxed{10}$.

Solution. Multiplying our polynomial by $x - 1$, we want $P(x) = x^n(x-2) + 1$ to have a root greater than 1.999. It's clear that such a root is less than 2. For $n \le 9$ and $x > 1.999$, $P(x) > -\frac{512}{1000} + 1 > 0$. When $n = 10$, $P(1.999) = -.001(2 - .001)^{10} + 1 < -.001[1024 - 512(10)(.001)] + 1 < -.018$, by Bernoulli's Inequality. Since $P(2) = 1$, it follows that P has a root greater than 1.999 by the Intermediate Value Theorem. So the smallest n which works is $n = 10$. $\square$

14. Let ABC be a triangle with $AB = 20$ and $AC = 18$. E is on segment AC and F is on segment AB such that $AE = AF = 8$. Let BE and CF intersect at G. Given that $AEGF$ is cyclic, then $BC = m\sqrt{n}$ for positive integers m and n such that n is not divisible by the square of any prime. Compute $100m + n$.

Proposed by James Lin.

Answer. $\boxed{305}$.

Solution. Solution 1: By angle chasing, (ABE) and (ACF) intersect on line BC. This intersection must be on segment BC; say it is at point X. Then

$$BC^2 = BC * BX + BC * XC = AB * BF + AC * CE = 20 * 12 + 18 * 10 = 420,$$

so $BC = 2\sqrt{105}$ and the answer is 305.

Solution 2: Let P and Q be the tangents of B to ω. Since C is the polar of B, points C, P, and Q are collinear. Let M, X and Y be the midpoints of segments BC, BP, and BQ, so M, X, Y are collinear as well. Let Ω be the circle centered at B with radius 0, so X and Y have equal power to circles ω and Ω, thus M lies on the radical axis of ω and Ω. Hence, $MB^2 = MO^2 - R^2 \implies BC^2 = 4MO^2 - 4R^2$. Since OM is a median of triangle BOC, $4MO^2 = 2OB^2 + 2OC^2 - BC^2$, so substituting gives $BC^2 = OB^2 + OC^2 - 2R^2 = P(B, \omega) + P(C, \omega)$. Proceed as in the previous solution.

Solution 3: Let ω be the circumcircle of $AEGF$, with radius R and center O. Let K be the foot of the altitude from B to OC. Since B is on the polar of C with respect to ω, $OK * OC = R^2$ or $OB * OC * \cos BOC = R^2$. Now, by the Law of Cosines on triangle BOC, $BC^2 = OB^2 + OC^2 - 2OB * OC \cos BOC = OB^2 - R^2 + OC^2 - R^2 = P(B, \omega) + P(C, \omega)$. Proceed as before. $\square$

15. Let $\mathbb{N}$ denote the set of positive integers. For how many positive integers $k \le 2018$ do there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = 2n$ for all $n \in \mathbb{N}$ and $f(k) = 2018$?

Proposed by James Lin.

Answer. $\boxed{1512}$.

Solution. We consider the positive integers as a family of chains; each chain is of the form $k \to 2k \to 4k \to 8k \to \cdots$ for every odd positive integer k. For any odd number a, we get that $f(a) = b$, $f(b) = 2a$, $f(2a) = 2b$, $f(2b) = 4a$, etc. so we must pair up chains. Furthermore, $\nu_2(b) \in \{0, 1\}$, since otherwise the odd number in the chain of b will not have anything to map to. Since we may pair chains however we like, the only restriction on k is that $\nu_2(k) \in \{0, 1\}$ except for $k = 1009, 2018$. This gives us a total of $1514 - 2 = 1512$ possibilities for k. $\square$

16. In a rectangular 57×57 grid of cells, k of the cells are colored black. What is the smallest positive integer k such that there must exist a rectangle, with sides parallel to the edges of the grid, that has its four vertices at the center of distinct black cells?

Proposed by James Lin.

Answer. $\boxed{457}$.

Solution. Label the rows $r_1, r_2, \ldots, r_{57}$. Then for such a rectangle to not exist, $\binom{57}{2} \geq \sum_{i=1}^{57} \binom{r_i}{2} \geq 57\binom{k/57}{2} \implies 56 \geq \frac{k}{57}\left(\frac{k}{57} - 1\right) \implies 8 \geq \frac{k}{57} \implies k \leq 456$, so $k \geq 457$ implies the existence of such a rectangle.

We construct $k = 456$ with no rectangle as follows: Label the rows and columns from 0 to 56 and denote the cell in column i and row j by (i, j). Color in the following cells for $0 \leq i \leq 7, 0 \leq j \leq 6$: $(49 + i, 7i + j), (7i + j, 49 + i), (56, 56)$. In the lower 49×49 grid, divide it into squares $S_{k,l}$ containing the cells in columns $7k, 7k+1, \ldots, 7k+6$ and in rows $7l, 7l+1, \ldots, 7l+6$ for $0 \leq k, l \leq 6$. In each square $S_{k,l}$, color the cells $(kl + m, m)$ modulo 7 for $0 \leq m \leq 6$. It's easy to verify that there is no rectangle with a vertex in rows 50 to 57 or columns 50 to 57. Any rectangle thus must have vertices in squares $S_{k_1,l_1}, S_{k_2,l_1}, S_{k_1,l_2}, S_{k_2,l_2}$ must satisfy $k_1(l_1 - l_2) \equiv k_2(l_1 - l_2) \pmod 7 \implies (k_1 - k_2)(l_1 - l_2) \equiv 0 \pmod 7$, which implies that $k_1 \equiv k_2 \pmod 7$ or $l_1 \equiv l_2 \pmod 7$, a contradiction.

Thus, the smallest k is $k = 457$. $\square$

17. Let S be the set of all subsets of $\{2, 3, \ldots, 2016\}$ with size 1007, and for a nonempty set T of numbers, let $f(T)$ be the product of the elements in T. Determine the remainder when

$$\sum_{T \in S} \left(f(T) - f(T)^{-1}\right)^2$$

is divided by 2017. Note: For b relatively prime to 2017, we say that b^{-1} is the unique positive integer less than 2017 for which 2017 divides $bb^{-1} - 1$.

Proposed by Tristan Shin.

Answer. $\boxed{2014}$.

Solution. Work modulo $p = 2017$. We have that

$$y^{p-1} - 1 = (y-1)(y-2)\cdots\left(y - \frac{p-1}{2}\right)\left(y - \frac{p+1}{2}\right)\cdots(y - (p-2))(y - (p-1))$$

$$= (y-1)(y-2)\cdots\left(y - \frac{p-1}{2}\right)\left(y + \frac{p-1}{2}\right)\cdots(y+2)(y+1) = \left(y^2 - 1^2\right)\left(y^2 - 2^2\right)\cdots\left(y^2 - \left(\frac{p-1}{2}\right)^2\right).$$

Let $x = y^2$. We get that

$$x^{\frac{p-1}{2}} - 1 = \left(x - 1^2\right)\left(x - 2^2\right)\cdots\left(x - \left(\frac{p-1}{2}\right)^2\right).$$

Squaring this, we have that

$$x^{p-1}-2x^{\frac{p-1}{2}}+1=\left(x-1^2\right)\left(x-2^2\right)\cdots\left(x-\left(\frac{p-1}{2}\right)^2\right)\left(x-\left(\frac{p-1}{2}\right)^2\right)\cdots\left(x-2^2\right)\left(x-1^2\right)$$
$$=\left(x-1^2\right)\left(x-2^2\right)\cdots\left(x-(p-1)^2\right).$$

Let K denote the sum of the product of the roots of $x^{p-1}-2x^{\frac{p-1}{2}}+1$, taken $\frac{p-1}{2}$ at a time. By Viete's Formula, we have $K=-2\left(-1\right)^{(p-1)-\frac{p-1}{2}}=-2$. On the other hand, this is the sum over all a in T of a^2, where T is the set of products of 1008 distinct elements of $\{1,2,\ldots,2016\}$.

Let a be a member of T with 1 included in the product. Denote by $\overline{a}$ the conjugate of a, which is the product of the 1008 distinct elements of $\{1,2,\ldots,2016\}$ *not* chosen for a. Note that $\overline{a}=\frac{2016!}{a}=-\frac{1}{a}$ by Wilson's Theorem (or by looking at the polynomial above). Then our sum for K is

$$\sum_t t^2+\overline{t}^2=\sum_t t^2+\frac{1}{t^2}$$

, where the t are all the members of T with 1 included in the product. But then we can just ignore the 1 and our set of t reduces to the set S. Thus,

$$\sum_{t\in S} t^2+\frac{1}{t^2}=-2$$

and thus

$$\sum_{t\in S}\left(t-\frac{1}{t}\right)^2=\sum_{t\in S} t^2-2+\frac{1}{t^2}=-2-2\left|T\right|.$$

It is easy to see that $|T|=\binom{2015}{1007}=\frac{1}{2}\binom{2016}{1008}$. However, we have that $\binom{2016}{1008}=\frac{2016!}{1\cdot2\cdot3\cdots1008\cdot1008\cdot1007\cdots2\cdot1}=\frac{2016!}{1\cdot2\cdot3\cdots1008\cdot(-1009)\cdot(-1010)\cdots(-2015)\cdot(-2016)}=\frac{2016!}{2016!(-1)^{1008}}=1$. Thus, $|T|=\frac{1}{2}$ and we have that

$$\sum_{t\in S}\left(t-\frac{1}{t}\right)^2=-2-2\cdot\frac{1}{2}=-3=2014.$$

$\square$

18. Suppose that a,b,c are real numbers such that $a<b<c$ and $a^3-3a+1=b^3-3b+1=c^3-3c+1=0$. Then $\frac{1}{a^2+b}+\frac{1}{b^2+c}+\frac{1}{c^2+a}$ can be written as $\frac{p}{q}$ for relatively prime positive integers p and q. Find $100p+q$.

Proposed by Michael Ren.

Answer. $\boxed{301}$.

Solution. Let $p=a+b+c=0, q=ab+bc+c=-3, r=abc=-1$. The cubic discriminant $(a-b)^2(b-c)^2(c-a)^2$ is given by $-4q^3-27r^2=81$ as $p=0$. This means that as $a>b>c$, $(a-b)(b-c)(c-a)=9$.

Note that $(a^2+b)(b^2+c)+(b^2+c)(c^2+a)+(c^2+a)(a^2+b)=a^2b^2+b^2c^2+c^2a^2+a^3+b^3+c^3+ab+bc+ca+ab^2+bc^2+ca^2$ and $(a^2+b)(b^2+c)(c^2+a)=a^2b^2c^2+abc+a^3b^2+b^3c^2+c^3a^2+ab^3+bc^3+ca^3$. By Vieta, $a^2b^2+b^2c^2+c^2a^2=q^2-2pr=q^2=9$, $a^3+b^3+c^3=p^3-3pq+3r=-3$, and $a^2b^2c^2+abc=r^2+r=0$.

For the rest, the key idea is to find the value of cyclic thing by taking the difference with the other cyclic thing. For example, $ab^2+bc^2+ca^2=\frac{ab^2+bc^2+ca^2+a^2b+b^2c+c^2a+ab^2+bc^2+ca^2-a^2b-b^2c-c^2a)}{2}=\frac{(a+b+c)(ab+bc+ca)-3abc+(a-b)(b-c)(c-a)}{2}=6$. The rest follow similarly. We have that $a^3b^2+b^3c^2+$

$c^3a^2 = \frac{(a+b+c)((ab+bc+ca)^2-abc(a+b+c))-abc(ab+bc+ca)-(ab+bc+ca)(a-b)(b-c)(c-a)}{2} = 12$ and $ab^3+bc^3+ca^3 = \frac{((a+b+c)^2-2(ab+bc+ca))(ab+bc+ca)-abc(a+b+c)+(a-b)(b-c)(c-a)(a+b+c)}{2} = -9$.

The value of the sum is $\frac{9-3-3+6}{12-9} = 3$, so the requested answer is 301.

Alternatively, you can evaluate the symmetric polynomials given by the sum and product of $\frac{1}{a^2+b} + \frac{1}{b^2+c} + \frac{1}{c^2+a}$ and $\frac{1}{a+b^2} + \frac{1}{b+c^2} + \frac{1}{c+a^2}$. $\square$

19. Let $P(x)$ be a polynomial of degree at most 2018 such that $P(i) = \binom{2018}{i}$ for all integer i such that $0 \le i \le 2018$. Find the largest nonnegative integer n such that $2^n \mid P(2020)$.

    Proposed by Michael Ren.

    Answer. $\boxed{6}$.

    Solution. We have by finite differences that $\displaystyle\sum_{i=0}^{2019} P(2019-i)(-1)^i\binom{2019}{i} = 0$. Thus, $P(2019) = \binom{2018}{2018}\binom{2019}{1} - \binom{2018}{2017}\binom{2019}{2} + \cdots + \binom{2018}{0}\binom{2019}{2019}$, which is the x^{2018} coefficient of $(1+x)^{2018}(1-x)^{2019} = (1-x^2)^{2018}(1-x)$. This is simply $-\binom{2018}{1009}$.

    Now, we have that $P(2020) = -\binom{2018}{1009}\binom{2020}{1} - \binom{2018}{2018}\binom{2020}{2} + \cdots - \binom{2018}{0}\binom{2020}{2020}$. This is $-2020\binom{2018}{1009}$ minus the x^{2020} coefficient of $(1+x)^{2018}(1-x)^{2020} = (1-x^2)^{2018}(1-x)^2$, which evaluates to $-2020\binom{2018}{1009} - \left(-\binom{2018}{1009} + \binom{2018}{1008}\right) = -2019\binom{2018}{1009} - \binom{2018}{1008}$.

    Now, write $\binom{2018}{1008} = \frac{1009}{1010}\binom{2018}{1009}$, so $P(2020) = -\left(2019 + \frac{1009}{1010}\right)\binom{2018}{1009}$. The multiplier has odd numerator and denominator $2 \cdot 505$, so we just want $\nu_2\left(\binom{2018}{1009}\right) - 1$. By Legendre, $\nu_2\left(\binom{2018}{1009}\right) = 2s_2(1009) - s_2(2018) = s_2(1009) = 7$, so $n = 6$. $\square$

20. Let ABC be a triangle with $AB = 7, BC = 5$, and $CA = 6$. Let D be a variable point on segment BC, and let the perpendicular bisector of AD meet segments AC, AB at E, F, respectively. It is given that there is a point P inside $\triangle ABC$ such that $\frac{AP}{PC} = \frac{AE}{EC}$ and $\frac{AP}{PB} = \frac{AF}{FB}$. The length of the path traced by P as D varies along segment BC can be expressed as $\sqrt{\frac{m}{n}}\sin^{-1}\left(\sqrt{\frac{1}{7}}\right)$, where m and n are relatively prime positive integers, and angles are measured in radians. Compute $100m+n$.

    Proposed by Edward Wan.

    Answer. $\boxed{240124}$.

    Solution. All angles in the below solution are in radians.

    We will start by showing a lemma, which will later prove helpful:

    Lemma. For any acute triangle ABC and a point $X \in BC$, let Y, Z be on AC, AB respectively so that $XY = CY$ and $XZ = BZ$. Then, $AYOZ$ is cyclic, where O denotes the circumcenter of $\triangle ABC$.

    Proof. This is a fairly simple length computation.

    Firstly, by symmetry, WLOG assume that X is on segment BU, where U is the foot of the altitude from A to BC. Let M_b, M_c be the midpoints of AC, AB, respectively. Then, by our assumption, Z is on segment BM_c and M_b is on segment CY. Note that $\angle M_cOM_b = \pi - \angle A$, so we just need $\triangle OM_cZ \sim \triangle OM_bY$ to show that $\angle YOZ = \pi - \angle A$. To show this, we just need $\frac{OM_c}{M_cZ} = \frac{OM_b}{M_bY}$, i.e.,

    $$OM_c * M_bY = OM_b * M_cZ.$$

    Note that $M_bY = YC - \frac{b}{2}$ and $M_cZ = \frac{c}{2} - ZB$. Furthermore, we know that $OM_c = R\cos\angle C$, $OM_b = R\cos\angle B$, $BZ = \frac{BX}{2\cos\angle B}$, and $CY = \frac{CX}{2\cos\angle C}$, where R is the circumradius of $\triangle ABC$. Plugging all of this in, we want that

    $$R\cos\angle C * \left(\frac{CX}{2\cos\angle C} - \frac{b}{2}\right) = R\cos\angle B * \left(\frac{c}{2} - \frac{BX}{2\cos\angle B}\right).$$

Expanding and rearranging, this is equivalent to

$$\frac{R}{2} * (BX + CX) = R(\frac{c\cos\angle B}{2} + \frac{b\cos\angle C}{2}),$$

i.e.,

$$\frac{a}{2} = \frac{c\cos\angle B + b\cos\angle C}{2}.$$

Note that $c\cos\angle B = BU$ and $b\cos\angle C = CU$, so we have that:

$$c\cos\angle B + b\cos\angle C = a,$$

and the lemma is proven. $\square$

Now, we will return to the problem at hand. Let's define O as the circumcenter of ABC. We claim that $P \in (\triangle BOC)$ at all times. It suffices just to show the reverse implication; that is, for $P \in (\triangle BOC)$ *and on or inside the triangle* with E, F as the feet of the angle bisectors of $\angle APC, \angle APB$ onto AC, AB, respectively, then the reflection D of A over EF is on segment BC.

Note that $\angle EPF = \frac{\angle APB + \angle APC}{2} = \frac{360 - \angle BPC}{2} = 180 - \angle A$, so that $AEPF$ is cyclic.

Consider an inversion with respect to A with arbitrary radius. For a point Q, we denote with Q' the image of Q under this inversion. Since $AEPF$ is cyclic, $P' \in E'F'$. Note that $\angle AB'P' = \angle APB = 2\angle APF = \angle AF'P'$. Therefore, $\angle B'P'F' = \angle AF'P'$, so $F'B' = B'P'$. Analogously, $E'C' = C'P'$. Therefore, by using the lemma in $\triangle AE'F'$ with point $P' \in E'F'$, we obtain that $AB'O_1C'$ is cyclic, where O_1 is the circumcenter of $\triangle AE'F'$. We claim that in fact $D' = O_1$. Let M be the midpoint of AD. Note that M is the foot of the altitude from A onto EF. Observe that A, D', and O_1 are collinear, due to the isogonal conjugacy of the circumcenter and the orthocenter, and the fact that $AE'F \sim AFE$. Let A' be the point diametrically opposite A on the circumcircle of $\triangle AE'F'$. Then, note that O_1 is the midpoint of AA'. Therefore, we just need to show that M inverts to A'. Since A, D', and O_1 are collinear, we know that A, M, and A' are collinear. Therefore, since $M \in EF$, we know that $AE'M'F'$ is cyclic, and so since $AE'A'F'$ is cyclic, $M' = A'$. Finally, we see that $D' = O_1$, as desired. Now, we have that $AB'D'C'$ is cyclic.

Inverting back, we find that $D \in BC$, as desired. We claim that this D is in fact on the segment BC. It suffices to show that ray AO_1 is on or inside $\angle E'AF'$. To show this, note that $\triangle AE'F' \sim \triangle AFE$. Furthermore, $\angle AEF = \angle APF = \frac{\angle APB}{2} \leq \frac{\pi}{2}$, and analogously $\angle AFE \leq \frac{\pi}{2}$. Also, $\angle A < \frac{\pi}{2}$ (easily verified with the Law of Cosines). Therefore, $\triangle AEF$ is non-obtuse, and so therefore contains it circumcenter. Thus, ray AO_1 is on or inside $\angle E'AF'$, and so D is on segment BC.

Now, let's check that D does indeed traverse every single point on BC as P traverses this arc of $(\triangle BOC)$. To do this, by continuity, we just need to check that it hits the endpoints, B and C. When P is the point on segment AB such that $AP = PC$, $D = C$. Similarly, when P is the point on segment AC such that $AP = PB$, $D = B$. Therefore, as P traverses this arc of $(\triangle BOC)$, D does indeed traverse every point on segment BC.

In conclusion, we see that the path traced by P is exactly the arc of $(\triangle BOC)$ which is inside $\triangle ABC$, so it suffices to find the length of this arc. Let $X \in AC$ and $Y \in AB$ be the endpoints of this arc. Note that since $\angle BXC = \angle BOC = 2\angle A$, we have $\angle ABX = \angle BXC - \angle A = \angle A$, so $AX = XC$. Therefore, $\angle XCY = \angle XCA = \angle A$, so the length of the arc is $\angle A * 2R'$, where R' denotes the circumradius of $\triangle BOC$. By the Extended Law of Sines in $\triangle BOC$, we know that $2R' = \frac{BC}{\sin\angle BOC} = \frac{5}{\sin 2\angle A}$. By the Law of Cosines, $\cos\angle A = \frac{b^2+c^2-a^2}{2bc} = \frac{6^2+7^2-5^2}{2*6*7} = \frac{60}{2*6*7} = \frac{5}{7}$. Therefore, as $\angle A$ is in $(0, \pi)$, we have that $\sin\angle A > 0$, so $\sin\angle A = \sqrt{1-\cos^2\angle A} = \frac{2\sqrt{6}}{7}$. By the Double Angle Formula, we get that $\sin 2\angle A = 2\sin\angle A * \cos\angle A = \frac{2*5*2\sqrt{6}}{7*7} = \frac{20\sqrt{6}}{49}$. Therefore, we have that

$$2R' = \frac{5}{\sin 2\angle A} = \frac{5}{\frac{20\sqrt{6}}{49}} = \frac{49}{4\sqrt{6}}.$$

Thus, the length of the path is

$$\frac{49}{4\sqrt{6}} * \angle A.$$

We claim that $\frac{\angle A}{2} = \sin^{-1}(\sqrt{\frac{1}{7}})$. To show this, note that $\cos\angle A = 1 - 2\sin^2(\frac{\angle A}{2})$, so that $\sin\frac{\angle A}{2} = \sqrt{\frac{1-\cos\angle A}{2}} = \sqrt{\frac{1-\frac{5}{7}}{2}} = \sqrt{\frac{1}{7}}$, as desired. Therefore, the length of the path is

$$\frac{49}{4\sqrt{6}} * \angle A = \frac{49}{2\sqrt{6}} * \frac{\angle A}{2} = \frac{49}{\sqrt{24}} * \sin^{-1}(\sqrt{\frac{1}{7}}) = \sqrt{\frac{2401}{24}} * \sin^{-1}(\sqrt{\frac{1}{7}}).$$

Our answer is therefore $100 * 2401 + 24 = 240124$. □

21. Let $\bigoplus$ and $\bigotimes$ be two binary boolean operators, i.e. functions that send $\{\text{True}, \text{False}\} \times \{\text{True}, \text{False}\}$ to $\{\text{True}, \text{False}\}$. Find the number of such pairs $(\bigoplus, \bigotimes)$ such that $\bigoplus$ and $\bigotimes$ distribute over each other, that is, for any three boolean values a, b, c, the following four equations hold:

    (a) $c \bigotimes (a \bigoplus b) = (c \bigotimes a) \bigoplus (c \bigotimes b)$;
    (b) $(a \bigoplus b) \bigotimes c = (a \bigotimes c) \bigoplus (b \bigotimes c)$;
    (c) $c \bigoplus (a \bigotimes b) = (c \bigoplus a) \bigotimes (c \bigoplus b)$;
    (d) $(a \bigotimes b) \bigoplus c = (a \bigoplus c) \bigotimes (b \bigoplus c)$.

    Proposed by Yannick Yao.

    Answer. $\boxed{22}$.

    Solution. For convenience we use 1 and 0 in this solution instead of True and False. Consider the following cases:

    Case 0: One of $\bigoplus$ or $\bigotimes$ is constant. WLOG assume $a \bigoplus b = 0$ for all a and b, then the four equations reduce to $c \bigotimes 0 = 0 \bigotimes c = 0 \bigotimes 0 = 0$, so the only possibilities are that $\bigotimes$ is either constant 0 or the AND operator ($1 \bigotimes 1 = 1$). It is easy to check that both possibilities are valid. This case accounts for 6 possible pairs: (0, 0), (0, AND), (AND, 0), (1, 1), (1, OR), (OR, 1).

    For the remaining cases, assume that neither operators is constant. Denote $p_{ab} = a \bigoplus b$ and $t_{ab} = a \bigotimes b$. From (1) and (2), we get that $a \bigotimes (b \bigoplus b) = (a \bigotimes b) \bigoplus (a \bigotimes b) = (a \bigoplus a) \bigotimes b$. (A similar identity can be obtained by swapping the place of the two operators.) We casework on p_{00} and p_{11}, in increasing order of complexity.

    Case 1: $p_{00} = 1, p_{11} = 0$. Then the previous identity gives $t_{00} = t_{11} = u$ (by letting $a = 0, b = 1$) and $t_{01} = t_{10} = v$ (by letting $a = b = 0$). By assumption $u \neq v$, so we assume WLOG that $(u, v) = (0, 1)$. Using the symmetric version of the identity, we get $p_{01} = p_{10} = p_{00} = 1$. However, we have that $(1 \bigoplus 0) \bigotimes 1 = 0 \neq 1 = (1 \bigotimes 1) \bigoplus (0 \bigotimes 1)$, so this case is impossible. (Similarly, we cannot have $t_{00} = 1, t_{11} = 0$.)

    Case 2: $p_{00} = p_{11}$. WLOG assume that $p_{00} = 0$, then from the previous identity we have $t_{00} = t_{01} = t_{10} = u$ (by letting $a = 0, b = 1$ and $a = 1, b = 0$) and $t_{11} = v \neq u$. From the previous case we see that $(u, v) \neq (1, 0)$, so we must have $u = 0, v = 1$ (in other words $\bigotimes$ is the AND operator). The equations (1) and (2) are guaranteed to be true whether $c = 0$ or $c = 1$, so we only need to care about the next two equations. Setting $a = c = 1, b = 0$ in equation (3) gives $p_{10} = 0$, and doing so in equation (4) gives $p_{01} = 0$. This forces $\bigoplus$ to be constant, so we may discard this case as well. (Similarly, we cannot have $t_{00} = t_{11}$.)

    Case 3: $p_{00} = 0, p_{11} = 1$. Similarly, we must also have $t_{00} = 0, t_{11} = 1$, which means that $\bigoplus, \bigotimes \in \{[a], [b], \text{AND}, \text{OR}\}$ ($[a]$ represent the operation where $a \bigoplus b = a$ for all a, b, and similar for $[b]$). Then since $a \bigoplus a = a = a \bigotimes a$ for all a, this renders all substitutions where $a = b$ into any of the four equations useless. Then if $a \neq b$, by Pigeonhole Principle either $a = c$ or $b = c$, and by making both substitutions to all four equations give the following *necessary and sufficient* conditions:

$$a \bigotimes (a \bigoplus b) = a \bigoplus (a \bigotimes b)$$
$$(a \bigoplus b) \bigotimes b = (a \bigotimes b) \bigoplus b$$
$$(a \bigoplus b) \bigotimes a = a \bigoplus (b \bigotimes a)$$
$$(b \bigotimes a) \bigoplus b = b \bigotimes (a \bigoplus b).$$

If $\bigoplus = [a]$, then it is not difficult to check that all four conditions are satisfied. The same applies when at least one of $\bigoplus$ and $\bigotimes$ is $[a]$ or $[b]$, so it remains to check when the two operators are both in $\{\text{AND}, \text{OR}\}$. However, it is well-known that AND and OR distribute over each other, and AND/OR distribute over themselves as well. Therefore, all $4^2 = 16$ pairs in this case are valid.

In conclusion, there are $6 + 16 = 22$ possible pairs in total. $\square$

22. Let $p = 9001$ be a prime number and let $\mathbb{Z}/p\mathbb{Z}$ denote the additive group of integers modulo p. Furthermore, if $A, B \subset \mathbb{Z}/p\mathbb{Z}$, then denote $A + B = \{a + b \pmod p) | a \in A, b \in B\}$. Let $s_1, s_2, \ldots, s_8$ are positive integers that are at least 2. Yang the Sheep notices that no matter how he chooses sets $T_1, T_2, \ldots, T_8 \subset \mathbb{Z}/p\mathbb{Z}$ such that $|T_i| = s_i$ for $1 \le i \le 8$, $T_1 + T_2 + \cdots + T_7$ is never equal to $\mathbb{Z}/p\mathbb{Z}$, but $T_1 + T_2 + \cdots + T_8$ must always be exactly $\mathbb{Z}/p\mathbb{Z}$. What is the minimum possible value of s_8?

Proposed by Yang Liu.

Answer. $\boxed{8856}$.

Solution. Let $k = 8$. I claim that for choice of $T_1, T_2, \ldots, T_{k-1}$, we can achieve the obvious upper bound of $|T_1 + T_2 + \cdots + T_{k-1}| = s_1 s_2 \ldots s_{k-1}$. The proof is somewhat of a "base-representation" construction. Specifically, for $1 \le i \le k-1$, define $p_i = \prod_{j=0}^{i-1} s_j$. Then we can let $T_i = \{0, p_i, 2p_i, \ldots, (s_i - 1)p_i\}$. Then it is clear that

$$T_1 + \cdots + T_{k-1} = \{0, 1, \ldots, s_1 s_2 \ldots s_{k-1} - 1\}$$

as desired.

For some other choice of $T_1, T_2, \cdots, T_k$, we can achieve $|T_1 + T_2 + \cdots + T_k| = s_1 + s_2 + \cdots + s_k - (k-1)$, which is a lower bound by Cauchy-Davenport. Therefore, $s_1 + s_2 + \cdots + s_k - (k-1) \ge p \implies s_k \ge p + (k-1) - (s_1 + s_2 + \cdots + s_{k-1})$.

So we want to maximize the sum of $s_1 + s_2 + \cdots + s_{k-1}$ given that $s_1 s_2 \ldots s_{k-1} < p$. A bit of checking yields that $s_1 = s_2 = \cdots = s_5 = s_6 = 2, s_7 = 140$ is the best. Then $s_8 = 9001 + 7 - 152 = 8856$. $\square$

23. Let ABC be a triangle with $BC = 13, CA = 11, AB = 10$. Let A_1 be the midpoint of BC. A variable line ℓ passes through A_1 and meets AC, AB at B_1, C_1. Let B_2, C_2 be points with $B_2B = B_2C, B_2C_1 \perp AB, C_2B = C_2C, C_2B_1 \perp AC$, and define $P = BB_2 \cap CC_2$. Suppose the circles of diameters BB_2, CC_2 meet at a point $Q \neq A_1$. Given that Q lies on the same side of line BC as A, the minimum possible value of $\frac{PB}{PC} + \frac{QB}{QC}$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c with $\gcd(a, c) = 1$ and b squarefree. Determine $a + b + c$.

Proposed by Vincent Huang.

Answer. $\boxed{773}$.

Solution. Let $A_2 = B_1C_2 \cap C_1B_2$. By Desargues' on $ABC, A_2B_2C_2$ we know AA_2, BB_2, CC_2 meet at P. By angle-chasing we have $\angle BPC = 180^\circ - \angle C_2CA_1 - \angle B_2BA_1 = \angle BB_2A_1 + \angle CC_2A_1 = \angle BC_1A_1 + \angle CB_1A_1 = \angle A$, hence $P \in (ABC)$. By similar reasoning $P \in (A_2B_2C_2)$.

Now clearly $A_2B_2C_2, ABC$ are directly similar and oriented $90°$ apart from one another. Let Q' be the intersection of $(A_2B_2C_2), (ABC)$ other than P; clearly since $P \in AA_2, BB_2, CC_2$ we know Q' is the center of a ninety degree spiral similarity sending $A_2B_2C_2$ to ABC. Then $\angle BQ'B_2 = \angle CQ'C_2 = 90°$, implying $Q = Q'$. But then if O, O_1 are the circumcenters of $ABC, A_2B_2C_2$ by similarity we have $\angle OQO_2 = 90°$, hence $(A_2B_2C_2), (ABC)$ are orthogonal. Now since B_2, C_2, O are collinear it follows that $(P, Q; B_2, C_2)$ is harmonic.

The spiral similarity sending $A_2B_2C_2$ to ABC must send P to a point P' on (ABC) with $\angle PQP' = 90°$, i.e. P' is the antipode to P. Then since $(P, Q; B_2, C_2)$ was harmonic it follows that $(P', Q; B, C)$ is as well. Now suppose $\angle BAP' = \theta_1, \angle CAP' = \theta_2$. Clearly since A, Q are on the same side of BC we have that P' is on the opposite side, hence P is on the same side of BC as Q as $\angle A$ is acute. So we deduce $\frac{QB}{QC} = \frac{P'B}{P'C} = \frac{\sin\theta_1}{\sin\theta_2}$, and $\frac{PB}{PC} = \frac{\sin(90°-\theta_1)}{\sin(90°-\theta_2)} = \frac{\cos\theta_1}{\cos\theta_2}$.

Then $\frac{PB}{PC} + \frac{QB}{QC} = \frac{\cos\theta_1\sin\theta_2+\sin\theta_1\cos\theta_2}{\cos\theta_2\sin\theta_2} = \frac{2\sin A}{\sin 2\theta_2} \geq \frac{2\sin A}{1}$. Equality is achieved at $\theta_2 = 45°$, which is possible since $\angle A > 45°$. To construct a valid diagram, we start with the position of P', which is on minor arc BC of (ABC) with $\angle CAP' = 45°$. Then we use P' to recover P, at which point we can define $B_2 = BP \cap A_1O, C_2 = CP \cap A_1O$, from which B_1, C_1 can be recovered to produce a line ℓ.

Anyway, the minimum value of the expression is $2\sin A$. Standard computations yield $\cos A = \frac{13}{55}$, hence $2\sin A = \frac{4\sqrt{714}}{55}$, yielding an answer of 773. $\square$

24. Find the number of ordered triples (a, b, c) of integers satisfying $0 \leq a, b, c \leq 1000$ for which

$$a^3 + b^3 + c^3 \equiv 3abc + 1 \pmod{1001}.$$

Proposed by James Lin.

Answer. $\boxed{622080}$.

Solution. Solution 1: First we solve the problem mod p, a prime ($p \neq 2, 3$). We show that there's $(p-1)^2$ solutions when $p \equiv 1 \pmod 3$ and $p^2 - 1$ solutions when $p \equiv 2 \pmod 3$.

We work mod p. Take an arbitrary triple (a, b, c) with $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq bc + ca + ab$, and look at triples $(a + S, b + S, c + S)$ for $S = 0, 1, 2, \ldots, p-1$. I claim that exactly one value of S gives a triple $(a + S, b + S, c + S)$ that satisfies the equation. Observe that the equation is equivalent to

$$(a + b + c + 3S)\left(a^2 + b^2 + c^2 - bc - ca - ab\right) = 1.$$

Then the only value of S that can work is

$$S = \frac{1}{3\left(a^2 + b^2 + c^2 - bc - ca - ab\right)},$$

so the claim is true.

There are p^2 total triples (a, b, c) with $a+b+c = 0$ (pick a, b arbitrarily then c is uniquely determined). So we just need to count when $a + b + c = 0$ and $a^2 + b^2 + c^2 = bc + ca + ab$. But since $a + b + c = 0$, we have $a^2 + b^2 + c^2 + 2bc + 2ca + 2ab = 0$, hence $bc + ca + ab = 0$. So a, b, c are the roots (mod p) of $X^3 - m$ for some m.

If $X^3 - m$ has a double root, then it is a root of its derivative, $3X^2$, so we would have $m = 0$. This works for one solution.

Otherwise, a, b, c must be the distinct roots of $X^3 - m$, $m \neq 0$. If $p \equiv 1 \pmod 3$, then there are $\frac{p-1}{3}$ values of m by primitive roots, but we can go with any of the 3! orders of a, b, c, so we have $2p - 2$ such (a, b, c) here. If $p \equiv 2 \pmod 3$, then there are no values of m since the third roots of unity do not exist mod p.

So when $p \equiv 1 \pmod 3$, there are $p^2 - 1 - (2p - 2) = (p-1)^2$ solutions to the equation, while when $p \equiv 2 \pmod 3$, there are $p^2 - 1$ solutions to the equation.

Since $1001 = 7 \cdot 11 \cdot 13$, we can CRT solutions together to get $6^2 \cdot 120 \cdot 12^2 = 622080$ solutions in total.

Solution 2: We work in $\mathbb{F}_p$ for $p \equiv 1 \pmod 3$, and in $\mathbb{F}_{p^2}$ for $p \equiv 2 \pmod 3$. In the $p \equiv 1 \pmod 3$ case, we take $\omega \neq 1 \in \mathbb{F}_p$ such that $\omega^3 = 1$ and factor the equation as

$$(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega) \equiv 1.$$

We take any of the $(p-1)^2$ possible nonzero x, y in F_p so that $x = a+b\omega+c\omega^2, y = a+b\omega^2+c\omega, \frac{1}{xy} = a+b+c$,. We can backsolve to get that $a = \frac{\frac{1}{xy}+x+y}{3}, b = \frac{\frac{1}{xy}+x\omega^2+y\omega}{3}, c = \frac{\frac{1}{xy}+x\omega+y\omega^2}{3}$, so all solutions work.

In the $p \equiv 2 \pmod 3$ case, we again take nonzero x, y in F_{p^2} such that $x = a+b\omega+c\omega^2, y = a+b\omega^2+c\omega, \frac{1}{xy} = a+b+c.$, so $3a = \frac{1}{xy}+x+y, 3b = \frac{1}{xy}+x\omega^2+y\omega, 3c = \frac{1}{xy}+x\omega+y\omega^2$ must all be in $\mathbb{F}_p$. This happens if and only if the following hold, by applying Frobenius Endomorphism:

$$\begin{cases} \frac{1}{x^py^p}+x^p+y^p = \frac{1}{xy}+x+y \\ \frac{1}{x^py^p}+x^p\omega+y^p\omega^2 = \frac{1}{xy}+x\omega^2+y\omega \\ \frac{1}{x^py^p}+x^p\omega^2+y^p\omega = \frac{1}{xy}+x\omega+y\omega^2 \end{cases}$$

Filtering these equations with ω gives us that this is equivalent to having $(xy)^{p-1} = 1$, $x^p = y$, $y^p = x$. Now note that any of the p^2-1 values of x will give a unique y that satisfies this set of equations, so there are p^2-1 total solutions.

Then we can proceed as in the previous solution. $\square$

25. Let m and n be positive integers. Fuming Zeng gives James a rectangle, such that $m-1$ lines are drawn parallel to one pair of sides and $n-1$ lines are drawn parallel to the other pair of sides (with each line distinct and intersecting the interior of the rectangle), thus dividing the rectangle into an $m \times n$ grid of smaller rectangles. Fuming Zeng chooses $m+n-1$ of the mn smaller rectangles and then tells James the area of each of the smaller rectangles. Of the $\binom{mn}{m+n-1}$ possible combinations of rectangles and their areas Fuming Zeng could have given, let $C_{m,n}$ be the number of combinations which would allow James to determine the area of the whole rectangle. Given that

$$A = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{C_{m,n}\binom{m+n}{m}}{(m+n)^{m+n}},$$

then find the greatest integer less than $1000A$.

Proposed by James Lin.

Answer. $\boxed{1289}$.

Solution. We consider the corresponding bipartite graph $K_{m,n}$ by giving a vertex to each of the m rows and n columns, a drawing an edge between two vertices if and only if the area of the corresponding rectangle is known. Now, note that if any three rectangle areas are known, where the rectangle centers form a right triangle with axes parallel to the whole rectangle, then the area of the fourth rectangle, whose center forms a rectangle with the other three centers, can clearly be determined. In our graph, what this means is that given any path of length 3, we can draw an edge between the two end vertices. We proceed by drawing edges between any vertices with distance three apart, until we no longer are able to.

Note that this process cannot decrease the number of connected components in the graph. Furthermore, any two vertices connected by a path will eventually become connected by an edge in this process. If the original $m+n-1$ edges form a tree, then clearly all edges must be drawn and the area of the whole rectangle can be determined. If there are $c > 1$ components remaining, then each of the components are $K_{a,b}$'s, the rows and columns can be rearranged so that James knows the areas of exactly c larger rectangles, say with sides $a_1, a_2, \ldots, a_c$ along one side of the whole rectangle and $b_1, b_2, \ldots, b_c$ along the other. It's clear that the value of $(a_1+a_2+\ldots+a_c)(b_1+b_2+\ldots+b_c)$ cannot be solely determined from the values of $a_1b_1, a_2b_2, \ldots, a_cb_c$ by varying a_1 and b_1 while keeping a_1b_1 and the rest of the a_i, b_i's fixed.

Thus, $C_{m,n}$ is the number of trees on a $K_{m,n}$, where the vertices of each side of the bipartite graph are distinguishable. We will show that $\sum_{m+n=S} C_{m,n}\binom{m+n}{m}$ is just twice the number of trees on a graph of S labeled vertices. In fact, consider any tree on S vertices. We can two-color the vertices in red and blue in two ways due to the lack of cycles, and these two colorings simply have red and blue colors swapped. If m vertices are colored red and n vertices are colored blue, and assign the red vertices to a "left half" of a $K_{m,n}$, and the blue vertices to a "right half." (So the second coloring of n red vertices and m blue vertices would be in a $K_{n,m}$ with n vertices on its left half and m on its right half.) For a given subset of size m of the S labeled vertices, it's clear that there will be $C_{m,n}$ graphs that get mapped to a $K_{m,n}$, so there will be $C_{m,n}\binom{m+n}{m}$ graphs on the S vertices for each $K_{m,n}$. This process is reversible, so by a bijection it follows that $\sum_{m+n=S} C_{m,n}\binom{m+n}{m} = 2(m+n)^{m+n-2}$ by Cayley's Theorem.

Thus, our sum simplifies to $A = \sum_{S\geq 2} \frac{2}{S^2}$, which is well-known to be $2\left(\frac{\pi^2}{6}-1\right)$. Using the fact that $3.1415 < \pi < 3.1416$, we get that $\lfloor 1000A \rfloor = 1289$.

Note: It's also a well-known fact that $C_{m,n} = m^{n-1}n^{m-1}$, although this is not necessary for this problem. $\square$

26. Let ABC be a triangle with incenter I. Let P and Q be points such that $IP \perp AC$, $IQ \perp AB$, and $IA \perp PQ$. Assume that BP and CQ intersect at the point $R \neq A$ on the circumcircle of ABC such that $AR \parallel BC$. Given that $\angle B - \angle C = 36^\circ$, the value of $\cos A$ can be expressed in the form $\frac{m-\sqrt{n}}{p}$ for positive integers m, n, p and where n is not divisible by the square of any prime. Find the value of $100m + 10n + p$.

Proposed by Michael Ren.

Answer. $\boxed{1570}$.

Solution. First we claim the locus of intersections of BP and CQ is the circumrectangular hyperbola of ABC passing through I.

Indeed, let $IP = IQ = x$. $x = 0$ tells us I lies on the locus and $x = \infty$ tells us the orthocenter H lies on the locus. It's well-known that any hyperbola through A, B, C, H is circumrectangular, so now it suffices to show that the locus is in fact a hyperbola. To do this, take three fixed values $x = a_1, a_2, a_3$ and let the corresponding points for P, Q be $(P_1, P_2, P_3), (Q_1, Q_2, Q_3)$. Let $R_i = BP_i \cap CQ_i$. Now consider a fourth variable value x and let the corresponding points for P, Q be P_x, Q_x; define R_x similarly to before. Clearly the cross ratios $(P_1, P_2; P_3, P_x), (Q_1, Q_2; Q_3, Q_x)$ are equal. It follows that $(BP_1, BP_2; BP_3, BP_x) = (BR_1, BR_2; BR_3, BR_x)$ and $(CP_1, CP_2; CP_3, CP_x) = (CR_1, CR_2; CR_3, CR_x)$ are equal. Since the cross ratio of $(R_1, R_2; R_3, R_x)$ is the same when viewed from B and C, we deduce that B, C, R_1, R_2, R_3, R_x lie on a conic. Therefore, as x varies, R_x lies on the fixed conic determined by B, C, R_1, R_2, R_3, and it's not hard to see this conic is a hyperbola. Hence, R is the isogonal conjugate of the infinity point on IO, implying that $AO \perp IO$. Letting O' be the reflection of O over AI, we have that $O'I \perp AO' \perp BC$, so $R + r = h_A$. Now this means $\cos A + \cos B + \cos C = 1 + r/R = h_A/R = 2\sin B \sin C$. Using the fact that $\cos A = -\cos(B+C)$, it follows that $\cos B + \cos C = \cos(B-C) = 2\cos\frac{B+C}{2}\cos\frac{B-C}{2}$. Using the fact that $\cos(B-C) = \cos 36^\circ = \frac{1+\sqrt{5}}{4}$ and $\cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}$, it follows quickly that $\cos A = -\cos(B+C) = \frac{15-\sqrt{5}}{20}$. Then the requested answer is 1570. $\square$

27. Let $n = 2^{2018}$ and let $S = \{1, 2, \ldots, n\}$. For subsets $S_1, S_2, \ldots, S_n \subseteq S$, we call an ordered pair (i, j) *murine* if and only if $\{i, j\}$ is a subset of at least one of S_i, S_j. Then, a sequence of subsets $(S_1, \ldots, S_n)$ of S is called *tasty* if and only if:

   (a) For all i, $i \in S_i$.

   (b) For all i, $\bigcup_{j \in S_i} S_j = S_i$.

   (c) There do not exist pairwise distinct integers $a_1, a_2, \ldots, a_k$ with $k \geq 3$ such that for each i, (a_i, a_{i+1}) is murine, where indices are taken modulo k.

   (d) n divides $1 + |S_1| + |S_2| + \ldots + |S_n|$.

   Find the largest integer x such that 2^x divides the number of tasty sequences $(S_1, \ldots, S_n)$.

   Proposed by Vincent Huang and Brandon Wang.

   Answer. $\boxed{2018}$.

   Solution. Draw a graph G and label its vertices $1, 2, ..., n$. We'll make G correspond to the S_i by drawing a directed edge from vertex i to vertex j for $i \neq j$ if and only if $j \in S_i$ (note that some vertices will have two edges drawn between them).

   Clearly $\sum_{j \in S_i} S_j$ contains S_i by condition 1), hence condition 2 is equivalent with $\sum_{j \in S_i} S_j \subseteq S_i$. So if $i \to j, j \to k$ in G, we require $i \to k$ as well, i.e. G is transitive.

   The third condition is clearly equivalent to the nonexistence of any undirected cycle in G of length at least 3. Therefore, if $i \to j$ and $j \to k$ for some $k \neq i$, we get a contradiction as condition 2 implies $i \to k$, which gives an undirected 3-cycle ijk. In other words, if $i \to j$, then j can't have outward edges except to i.

   So it's clear that any "2-cycles" $i \to j, j \to i$ have no edges going out of the cycle. We claim no edge can enter the cycle either. Indeed, suppose $k \to i$ for some $k \neq j$, so then $i \to j \implies k \to j$, which gives another undirected 3-cycle ijk, contradiction. Hence no edge can enter or leave a 2-cycle, so all 2-cycles are disconnected from the rest of the graph.

   Now suppose we have k such 2-cycles for $0 \leq k \leq 0.5n$. Then the remainder of the graph, which contains $n - 2k$ vertices, has no cycles at all, so the underlying undirected graph forms a forest. But clearly $\sum |S_i|$ equals $n + \sum \deg i$, where deg denotes the number of edges pointing out of a vertex, so we need $1 + \sum \deg i$ to be divisible by n. In particular, this sum must be at least n. But each of the $2k$ vertices in a 2-cycle contributes 1 to this sum, and the vertices in the forest together contribute at most $n - 2k - 1$ to the sum, so equality must hold, implying the forest is a tree.

   Now note that there are exactly $(n - 2k)^{n-2k-2}$ ways to draw the edges of this tree without direction by Cayley's Formula. Furthermore, if j is in the tree, we cannot simultaneously have $i \to j$ and $j \to k$ as that would imply $i \to k$ for a 3-cycle ijk, hence a vertex cannot have edges going in and out simultaneously. Now color the vertices of the tree in 2 colors such that no two vertices of the same color are adjacent- it's clear that one color must consist of vertices with all edges going outward and the other has all edges going inward, for a total of 2 ways to orient the edges. Hence there are $2(n - 2k)^{n-2k-2}$ to draw a valid configuration of edges within this tree.

   Now clearly the number of ways to pick k 2-cycles is $\binom{n}{2k} \frac{1}{k!} \binom{2k}{2,2,\ldots,2}$, so the expression we want is just $\sum_{0 \leq k \leq 0.5n} \binom{n}{2k} \frac{1}{k!} \binom{2k}{2, 2, ..., 2} \cdot 2(n - 2k)^{n-2k-2}$. For convenience let $n = 2m$. If $k = m$ then there are no vertices in the tree, so $\sum |S_i| = 2n$, a contradiction, hence k varies from 0 to $m - 1$.

   We now want to find the 2-adic valuation of $\sum_{k=0}^{m-1} \binom{2m}{2k} \binom{2k}{2, 2, ..., 2} \frac{1}{k!} (2m - 2k)^{2m-2k-2} \cdot 2$.

First we claim $\binom{2k}{2,2,..,2}\dfrac{1}{k!}$ is odd. Indeed, this expression is just $1 \cdot 3 ... \cdot (2k-1)$.

Now we'll replace k with $m-k$, so that we now want $\displaystyle\sum_{k=1}^{m}\binom{2m}{2k}\frac{1}{(m-k)!}\binom{2m-2k}{2,2,...,2}(2k)^{2k-2}\cdot 2$. I claim that for each k, we have $v_2\left(\binom{2m}{2k}\dfrac{1}{(m-k)!}\binom{2m-2k}{2,2,...,2}(2k)^{2k-2}\cdot 2\right) \geq 2018$ with equality only at $k=1$, which will imply the answer is 2018.

First, by the above claim, we can simplify our expression into $1+v_2\left(\binom{2m}{2k}\right)+(2k-2)v_2(2k)$. Clearly when $k=1$, we're left with $1+v_2(2^{2017}(2^{2018}-1)) = 2018$ as desired.

Suppose $k>1$. It's not hard to show that $v_2\left(\binom{2m}{2k}\right) = v_2\left(\binom{m}{k}\right) = 2017 - v_2(k)$ since m is a power of two. So we want $2018 - v_2(k) + (2k-2)v_2(2k) = 2018 + (2k-2) + (2k-3)v_2(k)$. If $k>1$, this expression is obviously at least $2018+1 = 2019$, hence the claim is proven.

So then the exponent of 2 that we want is 2018.

Remark: The number of tasty sequences is the number of categories with object set $[n]$ and $|\hom(a,b)| \leq 1$ such that

- for each a there is either exactly 1 arrow entering or exactly 1 exiting (id_a), and
- $|A| \equiv -1 \pmod{n}$.

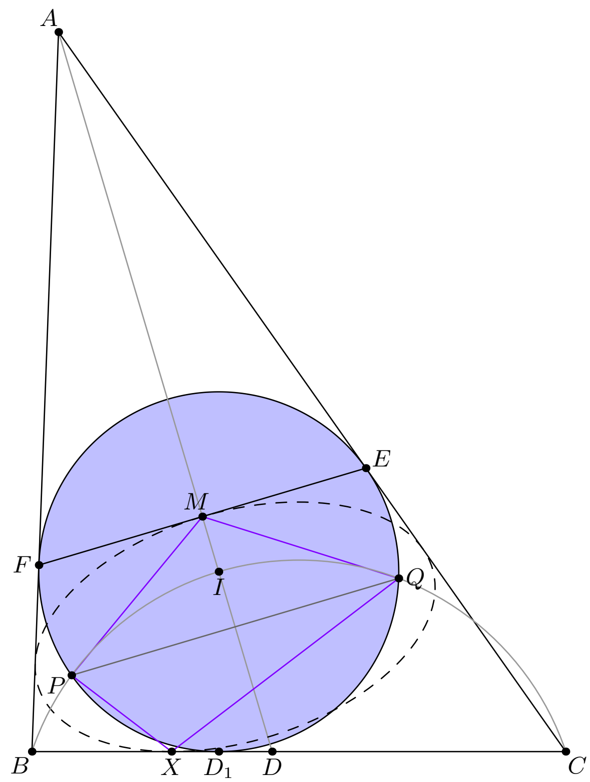
□

28. In $\triangle ABC$, the incircle ω has center I and is tangent to $\overline{CA}$ and $\overline{AB}$ at E and F respectively. The circumcircle of $\triangle BIC$ meets ω at P and Q. Lines AI and BC meet at D, and the circumcircle of $\triangle PDQ$ meets $\overline{BC}$ again at X. Suppose that $EF = PQ = 16$ and $PX + QX = 17$. Then BC^2 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m+n$.

Proposed by Ankan Bhattacharya and Michael Ren.

Answer. $\boxed{6764833}$.

Solution. Refer to the diagram below.



Without loss of generality, let E, F, P, and Q lie on ω in that order. It is clear that $\overline{EF} \parallel \overline{PQ}$ by the Incenter-Excenter Lemma, so since $EF = PQ$, it follows that $EFPQ$ is a rectangle, and P and Q are the antipodes of E and F on ω. Let M be the midpoint of $\overline{EF}$, and let ω touch $\overline{BC}$ at D_1.

Since B, C, I, P, Q are concyclic, by inverting in ω we observe that line PQ is the D_1-midline of $\triangle D_1EF$. Then I is the orthocenter of the medial triangle of $\triangle D_1EF$, and is the midpoint of the M-altitude. Since triangle D_1EF is homothetic to the excentral triangle of $\triangle ABC$ (which has orthocenter I), letting I_A denote the A-excenter, we deduce $AI = II_A$, so $AI_A = 2AI$. Thus the A-exradius of $\triangle ABC$ is twice the inradius, so $\frac{BC+CA+AB}{-BC+CA+AB} = 2$, or $AB + AC = 3BC$. Thus, BC is equal to the length of the tangent from A to ω.

Since line AI is the perpendicular bisector of $\overline{PQ}$, $DP = DQ$, so D is the midpoint of arc PQ containing X in $\odot(XPQ)$. It follows that $\angle BXP = \angle CXQ$, so there is an ellipse $\mathcal{E}$ with foci P and Q tangent to $\overline{BC}$ at D.

Now, note that $\overline{AP}$ and $\overline{AQ}$ are isogonal in $\angle BAC$ by symmetry, and $\angle BPC + \angle BQC = 2\angle BIC = 180^\circ + \angle A$. This implies that P and Q are isogonal conjugates in $\triangle ABC$, so $\mathcal{E}$ is tangent to $\overline{AC}$ and $\overline{AB}$ as well.

We contend that $\mathcal{E}$ is also tangent to $\overline{EF}$. Note $\angle BFQ = \angle EFP = 90^\circ$ and $\angle CEP = \angle FEQ = 90^\circ$, so P and Q are isogonal conjugates in $\triangle AEF$. Thus $\mathcal{E}$ is also tangent to $\overline{EF}$, and since $EFPQ$ is a rectangle, $\mathcal{E}$ is tangent to $\overline{EF}$ at M.

In particular $PM + QM = PX + QX = 17$, so $PM = QM = \frac{17}{2}$. Since $EM = FM = 8$, by the Pythagorean theorem the distance from M to $\overline{PQ}$ is $\frac{\sqrt{33}}{2}$, so $MI = \frac{\sqrt{33}}{4}$. It follows that $EI = FI = \frac{\sqrt{1057}}{4}$. Since $\triangle AEI \sim \triangle AME$ and $BC = AE$,

$$BC = AE = EI \cdot \frac{EM}{MI} = \frac{\sqrt{1057}}{4} \cdot \frac{8}{\frac{\sqrt{33}}{4}} = \frac{8\sqrt{1057}}{\sqrt{33}}.$$

Thus $BC^2 = \frac{67648}{33}$ and the answer is 6764833. $\square$

29. Let $q < 50$ be a prime number. Call a sequence of polynomials $P_0(x), P_1(x), P_2(x), ..., P_{q^2}(x)$ *tasty* if it satisfies the following conditions:

   - P_i has degree i for each i (where we consider constant polynomials, including the 0 polynomial, to have degree 0)
   - The coefficients of P_i are integers between 0 and $q-1$ for each i.
   - For any $0 \le i, j \le q^2$, the polynomial $P_i(P_j(x)) - P_j(P_i(x))$ has all its coefficients divisible by q.

   As q varies over all such prime numbers, determine the total number of tasty sequences of polynomials.

   Proposed by Vincent Huang.

   Answer. $\boxed{30416}$.

   Solution. We'll split into the cases $q > 2, q = 2$ and work in $\mathbb{F}_q[x]$. Case 1: $q > 2$. First it's not hard to show $P_1(x) = x$ for all such good sequences. Now transform $P_i(x) \to a^{-1}P(ax)$ for nonzero a so that P_2 is now monic. Since $p > 2$, we can also transform $P_i(x) \to P_i(x-a) + a$ so that $[x]P_2(x) = 0$. So now $P_2(x) = x^2 + c$ for some suitable values of c, and we have to remember to undo the $q(q-1)$ possible transformations at the end. Since $P_n(P_2(x))$ is even for each n, we must have $P_2(P_n(x))$ is also even, hence $P_n(x)^2$ is even, so P_n has the same parity as n for all n. Now if $n = 3$ this means $P_3(x) = x^3 + ax$. Solving $P_2(P_3(x)) = P_3(P_2(x))$ gives the solutions $(a, c) = (0, 0), (-3, -2)$. Next, note that P_n, P_2 commuting implies P_n is monic for each n. Let $P_n(x) = x^n + a_{n-1}x^{n-1} + ... + a_0$. I claim that there is a unique choice of a_i which allows P_n, P_2 to commute. Indeed, we have $(x^n + a_{n-1}x^{n-1} + ... + a_0)^2 + c = (x^2+c)^n + a_{n-1}(x^2+c)^{n-1} + ... + a_0$. Expansion of both sides and induction on i tells us that a_{n-i} is uniquely determined for each i. Now if $c = 0$ then $P_n = x^n$ works and is unique; if $c = 2$ then $P_n(2\cos\theta) = 2\cos n\theta$ works and is unique. It's not hard to check these are distinct solutions. But in the first case we can have $P_0 \equiv 0, 1$, while in the second case we are forced to have $P_0 \equiv 2$. Hence there are 3 total solutions, so $3q(q-1)$ total solutions before transforming. This yields 30408 total solutions after summing for $2 < q < 50$. Case 2: $q = 2$. I claim there are actually 8 solutions in this case. Indeed, P_3 and P_1 commuting tells us once again that $P_1(x) = x$. Unfortunately, this time the transform is useless as we can't "depress" the quadratic P_2 so we'll just have to do casework. Let $P_3(x) = x^3 + ax^2 + bx + c, P_2(x) = x^2 + ux + v$. Then P_3, P_2 commute, so $(x^3 + ax^2 + bx + c)^2 +$ deg 3 stuff $= (x^2 + ux + v)^3 +$ deg 4 stuff. On the left side, clearly the x^5 coefficient is zero, hence $u = 0$. If $v = 1$, then $(x^3 + ax^2 + bx + c)^2 + 1 = (x^2+1)^3 + a(x^2+1)^2 + b(x^2+1) + c$, so comparing x^4 coefficients yields a contradiction again. Therefore $P_2(x) = x^2$. Now we'll do the same thing for P_4. Since P_3, P_4 and the first term of P_3 is x^3, we have $(x^4 + ax^3 + bx^2 + cx + d)^3 +$ deg 8 stuff $= P_3(x)^4 +$ deg 9 stuff. By the Frobenius Endomorphism the right side is $x^{12} +$ deg 9 stuff. Therefore the x^{11}, x^{10} coefficients on the left side are equal, implying $a = b = 0$. Then the right side of the equation actually becomes deg 8 stuff, so comparing x^9 coefficients this time gives $c = 0$, hence $P_4(x) = x^4, x^4 + 1$. If $P_4(x) = x^4 + 1$ then we get a contradiction by comparing x^8 coefficients, so $P_4(x) = x^4$. Now again by the Frobenius Endomorphism, any choice of P_3 commutes with P_2, P_4. So we just list out the 8 possible values of P_3 and check which ones commute with the possible choices $P_0(x) = 0, 1$. Commuting with 0 requires $P_3(0) = 0$ so there are four solutions; commuting with 1 requires $P_3(1) = 1$ for four more solutions, hence 8 total. So the answer is $30408 + 8 = 30416$. $\square$

30. Let $p = 2017$. Given a positive integer n, an $n \times n$ matrix A is formed with each element a_{ij} randomly selected, with equal probability, from $\{0, 1, \ldots, p-1\}$. Let q_n be probability that $\det A \equiv 1 \pmod{p}$. Let $q = \lim_{n\to\infty} q_n$. If $d_1, d_2, d_3, \ldots$ are the digits after the decimal point in the base p expansion of q, then compute the remainder when $\sum_{k=1}^{p^2} d_k$ is divided by 10^9.

   Proposed by Ashwin Sah.

   Answer. $\boxed{98547790}$.

Solution. We consider matrices over $\mathbb{F}_p$. Notice that in a matrix A with $\det A \neq 0$ that we can, by multiplying the top row by a nonzero constant k, obtain each nonzero determinant exactly once. Without much difficulty we find that the number of matrices A with $\det A = 1$ is precisely $\frac{1}{p-1}$ of the number of matrices A with $\det A \neq 0$. This number reflects the amount of ways to choose n linearly independent vectors in $\mathbb{F}_p^n$. This clearly is just $(p^n-1)(p^n-p^1)\cdots(p^n-p^{n-1})$, since at each step there are precisely p^k vectors that are linearly dependent with the vectors already chosen. Divide by $p-1$ to get the amount we wish to count, and divide by p^{n^2} to obtain the desired probability:

$$q_n = \frac{1}{p-1}\prod_{k=1}^{n}\left(1-\frac{1}{p^k}\right)$$

Then

$$q = \frac{1}{p-1}\prod_{k=1}^{\infty}\left(1-\left(\frac{1}{p}\right)^k\right).$$

If we let $x = \frac{1}{p}$ then it becomes

$$q = \frac{x}{1-x}\prod_{k=1}^{\infty}\left(1-x^k\right).$$

Now, Euler's Pentagonal Number Theorem shows that the latter term expands as

$$1+\sum_{k=1}^{\infty}(-1)^k\left(x^{\frac{k(3k-1)}{2}}+x^{\frac{k(3k+1)}{2}}\right);$$

the former clearly expands as

$$x+x^2+x^3+\cdots$$

Let the coefficients of the resulting generating function in x be $a_1, a_2, \ldots$, noting that it has a zero constant term. Simple calculation shows that

$$a_k = \begin{cases} 1, & \text{for } k \in [6n^2+n+1, 6n^2+5n+1] \\ 0, & \text{for } k \in [6n^2+5n+2, 6n^2+7n+2] \\ -1, & \text{for } k \in [6n^2+7n+3, 6n^2+11n+5] \\ 0, & \text{for } k \in [6n^2+11n+6, 6n^2+13n+7] \end{cases}$$

for all integers $n \geq 0$.

Now since $x = \frac{1}{p}$, the base p expansion of q satisfies

$$d_k = \begin{cases} 1, & \text{for } k \in [6n^2+n+1, 6n^2+5n] \\ 0, & \text{for } k = 6n^2+5n+1 \\ p-1, & \text{for } k \in [6n^2+5n+2, 6n^2+7n+2] \\ p-2, & \text{for } k \in [6n^2+7n+3, 6n^2+11n+4] \\ p-1, & \text{for } k = 6n^2+11n+5 \\ 0, & \text{for } k \in [6n^2+11n+6, 6n^2+13n+7] \end{cases}$$

for all integers $n \geq 0$. Thus

$$\sum_{k=6n^2+n+1}^{6n^2+13n+7} d_k = n(6p-6)+(4p-6)$$

by simple summation, using the formula above over the appropriate ranges.

Now

$$p^2 = 2017^2 = 4068289 \in [6(823)^2+5(823)+2, 6(823)^2+7(823)+2] = [4068091, 4069737]$$

Thus our sum is

$$\begin{aligned}
&\sum_{n=0}^{822}(n(6p-6)+(4p-6))+4\cdot 823+(p-1)\cdot(2017^2-4068091+1)\\
&=(6p-6)\cdot\left(\frac{822\cdot 823}{2}\right)+(4p-6)\cdot 823+4\cdot 823+(p-1)\cdot 199\\
&=4098547790.
\end{aligned}$$

$\square$