The Online Math Open Spring Contest
Official Solutions
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1. Find the smallest positive integer that is relatively prime to each of $2, 20, 204$, and 2048.

   Proposed by Yannick Yao.

   Answer. $\boxed{1}$.

   Solution. 1 is the smallest positive integer. 1 is also relatively prime to every positive integer. Therefore the answer is 1. □

2. A positive integer n is called *bad* if it cannot be expressed as the product of two distinct positive integers greater than 1. Find the number of bad positive integers less than 100.

   Proposed by Michael Ren.

   Answer. $\boxed{30}$.

   Solution. If n is a prime number or 1, then it is certainly bad since it has no more than one factor greater than 1. If $n = p^2$ for some prime p, then the only ways to express it as a product of integers greater than 1 is $p \cdot p$, but the two numbers are not distinct. For all other cases of n, setting one factor to be the smallest prime divisor of n always work. Therefore, since there are 26 non-composites less than 100 (including 1) and 4 prime squares $(2^2, 3^2, 5^2, 7^2)$, there are $26 + 4 = 30$ bad positive integers less than 100. □

3. In rectangle $ABCD$, $AB = 6$ and $BC = 16$. Points P, Q are chosen on the interior of side AB such that $AP = PQ = QB$, and points R, S are chosen on the interior of side CD such that $CR = RS = SD$. Find the area of the region formed by the union of parallelograms $APCR$ and $QBSD$.

   Proposed by Yannick Yao.

   Answer. $\boxed{56}$.

   Solution. Suppose that AR and BS, BS and CP, CP and DQ, DQ and AR intersect at W, X, Y, Z respectively, then the quadrilateral $WXYZ$ is a rhombus, where $XZ = AP = \frac{AB}{3} = 2 = PQ$. This also implies that triangles PQY, XZY, ZXW, RSW are all congruent, and thus $WY = \frac{BC}{2} = 8$. The area of the union is therefore $2 \cdot 16 \cdot 2 - \frac{2 \cdot 8}{2} = 56$. □

4. Lunasa, Merlin, and Lyrica each has an instrument. We know the following about the prices of their instruments:

   - If we raise the price of Lunasa's violin by 50% and decrease the price of Merlin's trumpet by 50%, the violin will be \$50 more expensive than the trumpet;
   - If we raise the price of Merlin's trumpet by 50% and decrease the price of Lyrica's piano by 50%, the trumpet will be \$50 more expensive than the piano.

   Given these conditions only, there exist integers m and n such that if we raise the price of Lunasa's violin by $m\%$ and decrease the price of Lyrica's piano by $m\%$, the violin must be exactly $\$n$ more expensive than the piano. Find $100m + n$.

   Proposed by Yannick Yao.

   Answer. $\boxed{8080}$.

   Solution. Let V, T, P be the original price of the violin, trumpet, and the piano. We have $1.5V - 50 = 0.5T$ and $1.5T - 50 = 0.5P$, which gives $9V - 400 = P$, and we want $(1 + m\%)V - n = (1 - m\%)P$ for all V and P satisfying the previous relation. This requires $\frac{1+m\%}{1-m\%} = 9$, and solving the equation gives $m = 80$, and thus $n = \frac{400}{9}(1 + m\%) = 80$. Therefore we get $100m + n = 8080$. □

5. There are 15 (not necessarily distinct) integers chosen uniformly at random from the range from 0 to 999, inclusive. Yang then computes the sum of their units digits, while Michael computes the last three digits of their sum. The probability of them getting the same result is $\frac{m}{n}$ for relatively prime positive integers m, n. Find $100m + n$.

   Proposed by Yannick Yao.

   Answer. $\boxed{200}$.

   Solution. Suppose the first 14 integers and the last digit of the 15th integer has already been determined, so Yang's result is fixed and is less than 150, and is guaranteed to match Michael's sum in its unit digit. There's a $\frac{1}{100}$ probability that the first two digits of the 15th integer will make Michael's result match Yang's, so the answer is $100(1) + 100 = 200$. □

6. Let $ABCDEF$ be a regular hexagon with side length 10 inscribed in a circle ω. X, Y, and Z are points on ω such that X is on minor arc AB, Y is on minor arc CD, and Z is on minor arc EF, where X may coincide with A or B (And similarly for Y and Z). Compute the square of the smallest possible area of XYZ.

   Proposed by Michael Ren.

   Answer. $\boxed{7500}$.

   Solution. Suppose that O is the center of ω, and WLOG suppose that A, B, C, D, E, F are labeled clockwise. Let segments OX, OY, OZ intersect segments AB, CD, EF at X', Y', Z' respectively. Since none of $\angle XOY, \angle YOZ, \angle ZOX$ exceeds 180 degrees (when measured from X to Y to Z to X in clockwise direction), X', Y', Z' lie on the boundary or interior of XYZ and thus $[X'Y'Z'] \leq [XYZ]$. When two of X', Y', Z' are fixed, one can always slide the third point to an endpoint of the segment without increasing the area, which means that in order to minimize the area of $X'Y'Z'$, one can assume that all three vertices are vertices of the hexagon. Such triangles take on one of two forms: one is an equilateral triangle with side lengths $10\sqrt{3}$ and area $\frac{\sqrt{3}}{4}(10\sqrt{3})^2 = 75\sqrt{3}$, and the other is a right triangle with side lengths $10, 10\sqrt{3}, 20$ and area $\frac{1}{2}(10)(10\sqrt{3}) = 50\sqrt{3}$. Obviously the second one is smaller; therefore the minimum area of $X'Y'Z'$ is $50\sqrt{3}$ and consequently that is also the minimum area of XYZ, achieved when $X = X' = A, Y = Y' = C, Z = Z' = D$. So the answer is $(50\sqrt{3})^2 = 7500$. □

7. Let S be the set of all positive integers between 1 and 2017, inclusive. Suppose that the least common multiple of all elements in S is L. Find the number of elements in S that do not divide $\frac{L}{2016}$.

   Proposed by Yannick Yao.

   Answer. $\boxed{44}$.

   Solution. Since the highest powers of 2, 3, 7 below 2017 are $2^{10}, 3^6, 7^3$ respectively, the highest powers of 2, 3, 7 dividing $\frac{L}{2016}$ are $2^{10-5} = 2^5, 3^{6-2} = 3^4, 7^{3-1} = 7^2$ respectively. Therefore, those that do not divide $\frac{L}{2016}$ must be a multiple of $2^6 = 64, 3^5 = 243$, or $7^3 = 343$. Since a number between 1 and 2017 cannot be a multiple of two of the three numbers, we only need to count the 31 multiples of 64, 8 multiples of 243, and 5 multiples of 343, for $31 + 8 + 5 = 44$ numbers in total. □

8. A five-digit positive integer is called *k-phobic* if no matter how one chooses to alter at most four of the digits, the resulting number (after disregarding any leading zeroes) will not be a multiple of k. Find the smallest positive integer value of k such that there exists a k-phobic number.

   Proposed by Yannick Yao.

   Answer. $\boxed{11112}$.

Solution. When $k \le 10000$, each of the intervals $[10000, 19999], [20000, 29999], \dots, [90000, 99999]$ contains a multiple of 9 since each interval contains 10000 consecutive integers. When $10000 < k \le 11111$, these intervals contain $k, 2k, \dots, 9k$ respectively. Therefore, for any $k \le 11111$, there exists a multiple of k with any leading digit, so there does not exist a k-phobic number since one can keep the leading digit and change the rest to have a multiple of k. When $k = 11112$, the only multiples in range are $00000, 11112, 22224, 33336, 44448, 55560, 66672, 77784, 88896$, so we can see that 99959 is a 11112-phobic number, so 11112 is the smallest number that satisfies the condition. □

9. Kevin is trying to solve an economics question which has six steps. At each step, he has a probability p of making a sign error. Let q be the probability that Kevin makes an even number of sign errors (thus answering the question correctly!). For how many values of $0 \le p \le 1$ is it true that $p + q = 1$?

   Proposed by Evan Chen.

   Answer. $\boxed{2}$.

   Solution. Let $(a, b) = (p, 1-p)$. Then the desired probability is $\binom{6}{0}a^6 + \binom{6}{2}a^4b^2 + \binom{6}{4}a^2b^4 + \binom{6}{6}b^6 = \frac{1}{2}\left((a+b)^6 + (a-b)^6\right) = \frac{1}{2}\left(1 + (2p-1)^6\right)$. Setting this equal to $1-p$ gives $(2p-1)^6 = 1-2p$, which has $p = \frac{1}{2}$, $p = 0$ as real solutions. □

10. When Cirno walks into her perfect math class today, she sees a polynomial $P(x) = 1$ (of degree 0) on the blackboard. As her teacher explains, for her pop quiz today, she will have to perform one of the two actions every minute:

    - Add a monomial to $P(x)$ so that the degree of P increases by 1 and P remains monic;
    - Replace the current polynomial $P(x)$ by $P(x+1)$. For example, if the current polynomial is $x^2 + 2x + 3$, then she will change it to $(x+1)^2 + 2(x+1) + 3 = x^2 + 4x + 6$.

    Her score for the pop quiz is the sum of coefficients of the polynomial at the end of 9 minutes. Given that Cirno (miraculously) doesn't make any mistakes in performing the actions, what is the maximum score that she can get?

    Proposed by Yannick Yao.

    Answer. $\boxed{5461}$.

    Solution. Notice that the sum of coefficients is simply $P(1)$. Call the two actions type-(i) and type-(ii) respectively. It is not difficult to see that doing a type-(i) action on a degree-$(n-1)$ polynomial simply means adding the term x^n. Suppose that an x^n term is added before m type-(ii) actions, then this term will contribute $(m+1)^n$ to the value of $P(1)$. Therefore, among all strategies with k type-(i) actions and $9-k$ type-(ii) actions, the optimal one will have all the type-(i) actions precede the type-(ii) actions, and the maximal sum is $(10-k)^0 + (10-k)^1 + \cdots + (10-k)^k = \frac{(10-k)^{k+1}-1}{9-k}$, and we want to maximize this value over $k = 0, 1, \dots, 9$. When $k = 0, 1, \dots, 8$, the value of the RHS is equal to $1, 10, 73, 400, 1555, 3906, 5461, 3280, 511$ respectively. (Note that when $k = 9$, the value of the expression is 10 even though the RHS is not defined.) Therefore Cirno can get at most 5461 points for her pop quiz. □

11. Let a_1, a_2, a_3, a_4 be integers with distinct absolute values. In the coordinate plane, let $A_1 = (a_1, a_1^2)$, $A_2 = (a_2, a_2^2)$, $A_3 = (a_3, a_3^2)$ and $A_4 = (a_4, a_4^2)$. Assume that lines A_1A_2 and A_3A_4 intersect on the y-axis at an acute angle of θ. The maximum possible value for $\tan\theta$ can be expressed in the form $\dfrac{m}{n}$ for relative prime positive integers m and n. Find $100m + n$.

    Proposed by James Lin.

    Answer. $\boxed{503}$.

Solution. Consider two cases:

Case 1: the two lines intersect on the positive y-axis. Without loss of generality, assume that $a_1 < a_3 < 0 < a_2 < a_4$, and let $p = -a_1, q = a_2, r = -a_3, s = a_4$. It is not difficult to see that the line through $(-u, u^2)$ and (v, v^2) has slope $\frac{v^2-u^2}{v-(-u)} = v - u$, and intersect the y-axis at the point $(0, uv)$, which implies that $pq = rs$.

If both (or neither) of $p < q$ and $r < s$ are true, then both lines have positive (or negative) slopes, which means that $\theta < 45^\circ$.

Otherwise, we assume $p > q$ and $r < s$, and therefore $\theta = 180^\circ - \tan^{-1}(p - q) - \tan^{-1}(s - r)$. Since $\tan^{-1}(1) = 45^\circ$ and thus $\theta < 90^\circ$, we should make both $p - q$ and $s - r$ as small as possible. Since p, q, r, s are all distinct, it is not difficult to see that $|(p-q)-(s-r)| \geq 2$. Moreover, we can make one of them 1 and the other 4 by setting $(p, q, r, s) = (6, 2, 3, 4)$ (and $\theta > 45^\circ$), but $(p - q, s - r) = (1, 3)$ or $(3, 1)$ is impossible since it would imply that p, q, r, s are consecutive integers in some order, which cannot satisfy $pq = rs$. Therefore, the maximum possible value of $\tan\theta$ in this case is $-\frac{1+4}{1-1\cdot 4} = \frac{5}{3}$.

Case 2: the two lines intersect on the non-positive y-axis. In this case, we can see that all four points lie on the same side of y-axis, and so the slopes of both lines are both integers and have the same sign, and thus $\theta < 45^\circ$, which we need not consider since we have already found a bigger θ.

Therefore, the maximum possible value of $\tan\theta$ is $\frac{5}{3}$, and the answer is 503. □

12. Alice has an isosceles triangle M_0N_0P, where $M_0P = N_0P$ and $\angle M_0PN_0 = \alpha^\circ$. (The angle is measured in degrees.) Given a triangle M_iN_jP for nonnegative integers i and j, Alice may perform one of two *elongations*:

- an M-*elongation*, where she extends ray $\overrightarrow{PM_i}$ to a point M_{i+1} where $M_iM_{i+1} = M_iN_j$ and then removes the point M_i.
- an N-*elongation*, where she extends ray $\overrightarrow{PN_j}$ to a point N_{j+1} where $N_jN_{j+1} = M_iN_j$ and then removes the point N_j.

After a series of 5 elongations, k of which were M-elongations, Alice finds that triangle $M_kN_{5-k}P$ is an isosceles triangle. Given that 10α is an integer, compute 10α.

Proposed by Yannick Yao.

Answer. $\boxed{264}$.

Solution. For each triangle PM_iN_j, consider the ratio $\frac{\angle M_i}{\angle N_j} = \frac{m}{n}$. After one M-elongation, we see that $\angle M_{i+1} = \angle M_i/2$ and $\angle N_j' = \angle N_j + \angle M_i/2$, which means that $\frac{\angle M_{i+1}}{\angle N_j'} = \frac{m/2}{n+m/2} = \frac{m}{2n+m} = \frac{m}{n+(n+m)}$. Similarly, one N-elongation will change the ratio to $\frac{m+(m+n)}{n}$. Notice that since initially the ratio is $\frac{1}{1}$, and each elongation will double the sum of numerator and denominator, the sum of the numerator and denominator after t elongations is 2^{t+1} (and since both numerator and denominator are odd, this also implies that they will always be relatively prime). Moreover, if we express both m and n in binary, the t-th elongation will add a 1 at the $(t + 1) - th$ digit from the right on either m or n. It is not difficult to see that any pair of (m, n) that are both odd with sum 2^{t+1} is achievable with some combinations of M- and N- elongations.

After 5 elongations, the sum of the numerator and denominator is $2^6 = 64$, and since the final triangle is isosceles, $\angle P$ must be equal to one of the other two angles (since $m \neq n$ starting with the first elogation, $\angle M_i \neq \angle N_j$). Therefore, the sum of 64 and one of the m and n needs to be an odd factor of $10 \cdot 180 = 1800$, and since this sum is between $64 + 1 = 65$ and $64 + 63 = 127$, the only possibility is $64 + 11 = 75$, achieved when $(m, n) = (11, 53)$ or $(53, 11)$. Thus we can see that the desired ratio is $\alpha = 180 \cdot \frac{11}{75} = 26.4$ and therefore $10\alpha = 264$. □

13. On a real number line, the points $1, 2, 3, \dots, 11$ are marked. A grasshopper starts at point 1, then jumps to each of the other 10 marked points in some order so that no point is visited twice, before returning to point 1. The maximal length that he could have jumped in total is L, and there are N possible ways to achieve this maximum. Compute $L + N$.

Proposed by Yannick Yao.

Answer. $\boxed{28860}$.

Solution. It's not difficult to see that the longest length is $L = 2+4+6+8+10+10+8+6+4+2 = 60$, by counting the maximal number of times each unit segment gets covered. Moreover, one can show that each jump must either jump from one of the first 5 points to the last 5 points, or vice versa, or jumping to/from point 6. (The point before and after 6 must also belong to different sides). Using this, we may ultimately compute that the number of ways is $N = 2(5!)^2 = 28800$. Therefore the answer is $N + L = 28860$. $\square$

14. Let ABC be a triangle, not right-angled, with positive integer angle measures (in degrees) and circumcenter O. Say that a triangle ABC is *good* if the following three conditions hold:

- There exists a point $P \neq A$ on side AB such that the circumcircle of $\triangle POA$ is tangent to BO.
- There exists a point $Q \neq A$ on side AC such that the circumcircle of $\triangle QOA$ is tangent to CO.
- The perimeter of $\triangle APQ$ is at least $AB + AC$.

Determine the number of ordered triples $(\angle A, \angle B, \angle C)$ for which $\triangle ABC$ is good.

Proposed by Vincent Huang.

Answer. $\boxed{59}$.

Solution. The first two conditions imply that $\angle OBP = \angle BAO = \angle POB$, which means that BPO is similar to triangle BOA, and analogously triangle CQO is similar to COA. This requires that $AB, AC > R$ or equivalently, $\angle B, \angle C > 30°$.

We can also see that $BP = PO$ and similarly $CQ = QO$. Therefore $AB+AC = AP+PO+QO+AQ \geq AP + PQ + AQ$, so ABC is good if and only if equality holds, i.e. P, O, Q are collinear in that order. In particular, O is on or inside triangle ABC so ABC is acute or right.

But collinearity in this order occurs if and only if $\angle POB + \angle BOC + \angle COQ = 180°$, i.e. $(90° - \angle C) + 2\angle A + (90° - \angle B) = 180°$, which is equivalent to $\angle A = 60°$. Since $\angle B, \angle C > 30°$, $\angle B$ can range from $31°$ to $89°$, resulting in 59 good triangles. $\square$

15. Let $\phi(n)$ denote the number of positive integers less than or equal to n which are relatively prime to n. Over all integers $1 \leq n \leq 100$, find the maximum value of $\phi(n^2 + 2n) - \phi(n^2)$.

Proposed by Vincent Huang.

Answer. $\boxed{72}$.

Solution. It's well known that $\phi(n^2) = n\phi(n)$. When n is odd, $\phi(n^2 + 2n) = \phi(n)\phi(n+2)$, and when n is even, $\phi(n^2 + 2n) = 2\phi(n)\phi(n+2)$.

Let $f(n) = \phi(n^2 + 2n) - \phi(n^2)$. Then when n is even, $f(n) = \phi(n)[2\phi(n+2) - n]$. Since $n+2$ is even, $\phi(n+2) \leq \frac{n+2}{2}$ with equality if and only if $n+2$ is a power of 2. When $n+2$ is not a power of 2, then $f(n) \leq 0$. When $n+2$ is a power of 2, $f(n) = 2\phi(n) \leq n$. Since $n+2 \leq 102$, the largest power of 2 that $n+2$ can obtain is 64, giving $f(n) \leq 62$.

When n is odd, $f(n) = \phi(n)[\phi(n+2) - n]$. Note that $\phi(n+2) \leq n+1$ with equality if and only if $n+2$ is a prime. When $n+2$ is not a prime, then $f(n) \leq 0$. When $n+2$ is a prime, $f(n) = \phi(n) \leq n$.

Note that when $n+2 = 101, 97, 89, 83, 79$, we get that $f(n) = 60, 72, 56, 54, 60$, respectively. Otherwise, $n + 2 \le 73 \implies f(n) \le 71$.

Hence the maximum value of $f(n)$ is achieved when $n = 95$, giving an answer of 72. □

16. Let S denote the set of subsets of $\{1, 2, \ldots, 2017\}$. For two sets A and B of integers, define $A \circ B$ as the *symmetric difference* of A and B. (In other words, $A \circ B$ is the set of integers that are an element of exactly one of A and B.) Let N be the number of functions $f : S \to S$ such that $f(A \circ B) = f(A) \circ f(B)$ for all $A, B \in S$. Find the remainder when N is divided by 1000.

Proposed by Michael Ren.

Answer. $\boxed{112}$.

Solution. Consider each element A of S as a 2017-dimensional vector v_A with entries in $\mathbb{F}_2$, such that the ith entry of v_A equal to 1 if $i \in A$ and 0 otherwise. Define w_A similarly with respect to $f(A)$. Note that we have the condition $w_{A+B} = w_{A \circ B} = w_A \circ w_B = w_A + w_B$, so the problem now becomes determining the number of linear maps on $\mathbb{F}_2^{2017}$. By setting $A = B = \varnothing$, we have $w_\varnothing = w_{\varnothing+\varnothing} = w_\varnothing + w_\varnothing = \mathbf{0}$, so the empty set must map to itself. Moreover, the vectors $v_{\{1\}}, v_{\{2\}}, \ldots, v_{\{2017\}}$ are the basis of $\mathbb{F}_2^{2017}$, so one can assign each of $w_{\{1\}}, w_{\{2\}}, \ldots, w_{\{2017\}}$ to any one of the 2^{2017} vectors which also determines all other mappings consistently (because of linearity). Thus there are $(2^{2017})^{2017} = 2^{2017^2}$ possible functions, and we can reduce $2^{2017^2} \equiv 2^{89} \equiv (2^7)^{12} \cdot 2^5 \equiv 3^{12} \cdot 32 \equiv 112 \pmod{125}$ and thus $2^{2017^2} \equiv 112 \pmod{1000}$.

A more combinatorial way to phrase this solution would be to note that empty set must go to empty set by setting A and B to be the empty set and that setting the outputs of $\{1\}, \{2\}, \ldots, \{2017\}$ uniquely determines the entire function. □

17. Let ABC be a triangle with $BC = 7, AB = 5$, and $AC = 8$. Let M, N be the midpoints of sides AC, AB respectively, and let O be the circumcenter of ABC. Let BO, CO meet AC, AB at P and Q, respectively. If MN meets PQ at R and OR meets BC at S, then the value of OS^2 can be written in the form $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $100m + n$.

Proposed by Vincent Huang.

Answer. $\boxed{240607}$.

Solution. By the Law of Cosines, $\angle A = 60^\circ$. Since $\angle BOC = 120^\circ = 180^\circ - A$ we know A, P, O, Q are concyclic. Then the Simson line of O with respect to triangle APQ must be line MN, which meets PQ at R, implying $OR \perp PQ$.

Now define S' as the point on BC with B, S', O, Q concyclic. By Miquel's Theorem on triangle ABC and points S', P, Q, we know that C, P, O, S' are concyclic as well. It's easy to see $\angle QS'O = \angle QBO = \angle OAQ = \angle OPQ = 90^\circ - \angle C$ and similarly we deduce $90^\circ - \angle B = \angle PQO = \angle PS'O$, implying O is the orthocenter of triangle PQS', hence $OS' \perp PQ$. Therefore $S' = S$. Furthermore, from the angle-chasing we know that $\triangle SPQ \sim \triangle ABC$.

Let H be the orthocenter of $\triangle ABC$. Since $\angle A = 60^\circ$ it's well-known and easy to prove that $AH = AO$. Therefore by the similarity, OS is equal to the circumradius of $\triangle PQS$.

Let R, R' be the circumradii of triangles ABC, POQ. Since O is the orthocenter of PQS we know that $(POQ), (PSQ)$ are congruent, so it suffices to find the circumradius of $(APOQ)$. By the Law of Sines in $\triangle APO$, we know that $R = AO = 2R' \sin \angle APO = 2R' \sin(C + 30^\circ)$.

By standard methods we can compute $[ABC] = 10\sqrt{3}, R = \frac{7}{\sqrt{3}}, \cos C = \frac{11}{14}, \sin C = \frac{5\sqrt{3}}{14}$, so it's not hard to find $OS^2 = \frac{2401}{507}$, yielding an answer of 240607. □

18. Let p be an odd prime number less than 10^5. Granite and Pomegranate play a game. First, Granite picks a integer $c \in \{2, 3, \ldots, p-1\}$. Pomegranate then picks two integers d and x, defines $f(t) = ct + d$, and writes x on a sheet of paper. Next, Granite writes $f(x)$ on the paper, Pomegranate writes $f(f(x))$, Granite writes $f(f(f(x)))$, and so on, with the players taking turns writing. The game ends when two numbers appear on the paper whose difference is a multiple of p, and the player who wrote the most recent number wins. Find the sum of all p for which Pomegranate has a winning strategy.

Proposed by Yang Liu.

Answer. $\boxed{65819}$.

Solution. Let's say that c, d are already chosen. Let f_0 be the sequence defined by $f_0 = x$ and $f_{i+1} = cf_i + d$. Then $f_i = -\frac{d}{c-1} + \left(x + \frac{d}{c-1}\right) \cdot c^i$. To prevent losing, Pomegranate would of course first choose and $x \neq -\frac{d}{c-1}$. (Or else f_i is a constant sequence).

Otherwise, the sequence f_i repeats with period equal to $\text{ord}_p(c)$. So for Granite to win, he needs $\text{ord}_p(c)$ to be odd. Since $p > c > 1$ (a condition), we need for $p - 1$ to have an odd factor > 1. This happens unless p is a Fermat prime. So the sum of all possible primes is $3 + 5 + 17 + 257 + 65537 = 65819$. $\square$

19. For each integer $1 \leq j \leq 2017$, let S_j denote the set of integers $0 \leq i \leq 2^{2017} - 1$ such that $\left\lfloor \frac{i}{2^{j-1}} \right\rfloor$ is an odd integer. Let P be a polynomial such that

$$P\left(x_0, x_1, \ldots, x_{2^{2017}-1}\right) = \prod_{1 \leq j \leq 2017} \left(1 - \prod_{i \in S_j} x_i\right).$$

Compute the remainder when

$$\sum_{\left(x_0, \ldots, x_{2^{2017}-1}\right) \in \{0,1\}^{2^{2017}}} P\left(x_0, \ldots, x_{2^{2017}-1}\right)$$

is divided by 2017.

Proposed by Ashwin Sah.

Answer. $\boxed{1840}$.

Solution. First of all, the set S_j is exactly the set of all integers in $[0, 2^{2017} - 1]$ whose j-th rightmost digit in binary is odd. The value of P is equal to 1 if and only if for each j, there is at least one integer $i \in S_j$ such that $x_i = 0$. If we consider a bijection between all the integers in $[0, 2^{2017} - 1]$ with the set of all subsets $T_0, T_1, \ldots, T_{2^{2017}-1}$ of $T = \{1, 2, \ldots, 2017\}$ such that $j \in T_i$ if and only if $i \in S_j$, and consider each tuple $(x_0, x_1, \ldots, x_{2^{2017}-1})$ as a way of choosing a subset of $\{T_0, T_1, \ldots, T_{2^{2017}-1}\}$ (where 0 correspond to chosen and 1 correspond to not chosen), then for P to be equal to 1, the union of these chosen subsets must be equal to T itself. Therefore it suffices to count the number of ways to pick such a collection of subsets.

The number of ways to pick a collection of subsets whose union is a subset of a fixed $(2017-s)$-element subset is equal to $2^{2^{2017-s}}$. So by PIE, we find that the number of ways is $\sum_{s=0}^{2017} (-1)^s \binom{2017}{s} 2^{2^{2017-s}} \equiv 2^{2^{2017}} - 2 \pmod{2017}$. We notice $2^{2017} \equiv 2^7 \pmod{2016}$ so that we find $2^{2^7} - 2 \equiv 1840 \pmod{2017}$. $\square$

20. Let n be a fixed positive integer. For integer m satisfying $|m| \leq n$, define $S_m = \displaystyle\sum_{\substack{i-j=m \\ 0 \leq i,j \leq n}} \frac{1}{2^{i+j}}$. Then

$$\lim_{n \to \infty} \left(S_{-n}^2 + S_{-n+1}^2 + \ldots + S_n^2\right)$$

can be expressed in the form $\dfrac{p}{q}$ for relatively prime positive integers p, q. Compute $100p + q$.

Proposed by Vincent Huang.

Answer. $\boxed{8027}$.

Solution. Let $a_i = \dfrac{1}{2^i}$. We wish to consider the expression $(a_0a_n)^2 + (a_0a_{n-1} + a_1a_n)^2 + ... + (a_{n-1}a_0 + a_na_1)^2 + (a_na_0)^2$.

Each parenthesis consists of terms of the form a_ia_j with $i - j$ fixed. So if we expand, we get something of the form $\sum_{i-j=k-l} a_ia_ja_ka_l$. The key observation is that we can also write this sum in the form $\sum_{i+l=j+k} a_ia_ja_ka_l$, and grouping these terms by the value of $i + l = j + k$, the expression becomes $(a_0a_0)^2 + (a_0a_1 + a_1a_0)^2 + ... + (a_{n-1}a_n + a_na_{n-1})^2 + (a_na_n)^2 = \dfrac{1}{4^0} + \dfrac{4}{4^1} + ... + \dfrac{(n+1)^2}{4^n} + \dfrac{n^2}{4^{n+1}} + \dfrac{(n-1)^2}{4^{n+2}} + ... + \dfrac{1}{4^{2n}}$.

As n approaches infinity, this sum approaches $\sum_{i\geq 0} \dfrac{(i+1)^2}{4^i}$, which evaluates to $\dfrac{80}{27}$ by standard methods, so the answer is 8027. $\square$

21. Let $\mathbb{Z}_{\geq 0}$ be the set of nonnegative integers. Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a function such that, for all $a, b \in \mathbb{Z}_{\geq 0}$:
$$f(a)^2 + f(b)^2 + f(a+b)^2 = 1 + 2f(a)f(b)f(a+b).$$
Furthermore, suppose there exists $n \in \mathbb{Z}_{\geq 0}$ such that $f(n) = 577$. Let S be the sum of all possible values of $f(2017)$. Find the remainder when S is divided by 2017.

Proposed by Zack Chroman.

Answer. $\boxed{1191}$.

Solution. Note that $P(0,0) \implies f(0) = 1$. Then, letting $f(1) = k$, $P(1,m) \implies f(m+1)^2 - 2kf(m)f(m+1) = 1 - k^2 - f(m)^2$. By $P(1, m-1)$, this quadratic is satisfied by $f(m-1)$, so either $f(m+1) = f(m-1)$ or $f(m+1) = 2kf(m) - f(m-1)$. If $f(2) = 1$, $f(3)$ is k in both cases, and iterating this we see that the function goes $1, k, 1, k, 1, k, \ldots$. It turns out this function works, so we can have $f(2017) = 577$ by taking $k = 577$.

Otherwise, we have $f(2) = 2k^2 - 1$. Then, $f(3) \in \{k, 4k^3 - 3k\}$. If $f(3) = k$, $f(4)$ is one of $2k^2 - 1$ and 1. However, $P(2,2) \implies f(4) = 1$ or $f(4) = 2f(2)^2 - 1 = 8k^4 - 8k^2 + 1$. If this is also $2k^2 - 1$, solving the resulting quadratic gives $k = \pm 1$, but then $f(2) = 1$ which puts us back in the first case. Then we take $f(4) = 1$. Then we immediately get $f(5) = f(3) = k$, and $P(4,2)$ tells us that $f(6) = 2k^2 - 1$, not 1. In general, we get f repeats the sequence $\{1, k, 2k^2 - 1, k\}$. Noting that $577 = 2 \cdot 17^2 - 1$, and $2017 \equiv 3 \pmod 4$, this gives the possible value of 17 for $f(2017)$.

In the final case, we take $f(3) = 4k^3 - 3k$. I claim that from here on out, $f(n)$ is defined as $f(n) = 2kf(n-1) - f(n-2)$. This is clear upon computing $P(1, n-1) - P(2, n-2)$, which is linear in $f(n)$. It remains to determine which of these sequences 577 belongs to. We already know that $k = 17$ works, and clearly $k = 577$ works. It turns out $f(4) = 577$ when $k = 3$, and these are all the cases. Noting that $f(n) = \frac{1}{2}\left((3+2\sqrt{2})^n + (3-2\sqrt{2})^n\right)$ gives $f(2017) \equiv 3 \pmod{2017}$, and similarly when $k = 17$ and $k = 577$, $f(2017) \equiv 17 \pmod{2017}$ and $f(2017) \equiv 577 \pmod{2017}$ respectively. But in this case $f(2017) \neq 17$ or 577, so they need to be included separately. The sum of all answers modulo 2017 is $577 + 17 + (3 + 17 + 577) = 1191$. $\square$

22. Let $S = \{(x, y) \mid -1 \le xy \le 1\}$ be a subset of the real coordinate plane. If the smallest real number that is greater than or equal to the area of any triangle whose interior lies entirely in S is A, compute the greatest integer not exceeding $1000A$.

Proposed by Yannick Yao.

Answer. $\boxed{5828}$.

Solution. It is clear that the boundary of S is composed of two hyperbolas, namely $xy = 1$ and $xy = -1$. For ease of reference, we call the four branches of the two hyperbolas by the quadrant number they lie in (i.e. branch 1, branch 2, etc.).

Note that the affine transformation $(x, y) \to (kx, y/k)$ preserves both hyperbola and any area for any positive number k. Also note that the triangle with maximal area necessarily has at least one side tangent to the boundary of S (otherwise we would have three vertices on three different branches, and we could always slide one of them away from the opposite side, increasing the area until one side is tangent), so we may assume that the side is tangent at $T(1, 1)$ (after rotation and the aforementioned transformation).

The line $x + y = 2$ intersects branch 2 and 4 at points $P(1-\sqrt{2}, 1+\sqrt{2})$ and $Q(1+\sqrt{2}, 1-\sqrt{2})$ respectively, and if we let $R = (-1, -1)$, we may see that PR and QR are also tangent to branch 2 and branch 4 at P and Q respectively. From this, we may deduce that if we choose P' on the interior of PT and Q' on the interior of QT, then the tangent line to branch 2 through P' and the tangent line to branch 4 through Q' always intersect *outside* S (in Quadrant 3). As a result, we can always choose a point R' on branch 3 such that either $R'P'$ is tangent to branch 2 or $R'Q'$ is tangent to branch 4. WLOG assume the former case is true, then we can make sure that R' is to the left of R, and so either (1) we can move Q' to Q and get a larger area, or (2) in the process of doing (1), we got stuck at a point where $R'Q'$ is tangent to branch 3.

We show that (2) is impossible. In fact, if we do the affine transformation to make the tangent point on branch 2 $(-1, 1)$, then similar to the argument in the preceeding paragraph, we can show that Q' needs to be outside S since it's the intersection of two tangent lines, which contradicts the assumption of (2) itself.

Therefore, a maximal triangle necessarily have two vertices on the adjacent branches. With this in mind, we restart and WLOG let these two vertices be $A(a, 1/a)$ (on branch 1) and $B(b, -1/b)$ (on branch 4). Notice that the tangent line to branch 4 through A has equation $y = \frac{3+2\sqrt{2}}{a^2}(x-a) + \frac{1}{a}$, which intersects branch 4 at $((\sqrt{2}-1)a, -(\sqrt{2}+1)/a)$, implying that $\frac{b}{a} \ge \sqrt{2}-1$. Similarly we can show that $\frac{a}{b} \ge \sqrt{2}-1$. Therefore, we have $\frac{a^2+b^2}{ab} = \frac{a}{b} + \frac{b}{a} \in [2, 2\sqrt{2}]$.

It is obvious that the maximal triangle when points A and B are fixed has the third vertex C being the intersection of the line tangent to branch 2 through A and the line tangent to branch 3 through B. (If we place C to the right of AB where AC and BC are tangent to branch 1 and branch 4 respectively, then we may extend rays CA and CB to intersect branch 2 and branch 3 at A' and B' respectively and get a larger area.) Since the equations of the two tangent lines are $y = \frac{3-2\sqrt{2}}{a^2}(x-a) + \frac{1}{a}$ and $y = \frac{-3+2\sqrt{2}}{b^2}(x-b) - \frac{1}{b}$, we get that the intersection is

$$C = (-(2+2\sqrt{2})\frac{ab(a+b)}{a^2+b^2}, (2-2\sqrt{2})\frac{b-a}{a^2+b^2}).$$

And the area is equal to

$$\frac{1}{2}((b-x_C)(\frac{1}{a}-y_C) - (a-x_C)(-\frac{1}{b}-y_C)) = \frac{1}{2}(\frac{a^2+b^2}{ab} + \frac{8ab}{a^2+b^2} + 4\sqrt{2}).$$

Due to the result we established in the previous paragraph, it is not difficult to see that the right-hand side is maximized when $\frac{a^2+b^2}{ab} = 2$, or when $a = b$, and the maximum is $\frac{1}{2}(2 + \frac{8}{2} + 4\sqrt{2}) = 3 + 2\sqrt{2}$. This maximum can be achieved with the triangle whose vertices are at $A = (1, 1), B = (1, -1)$, and $C = (-2-2\sqrt{2}, 0)$. Since $3 + 2\sqrt{2} \approx 5.8284$, the final answer is 5828. $\square$

23. Determine the number of ordered quintuples (a, b, c, d, e) of integers with $0 \le a < b < c < d < e \le 30$ for which there exist polynomials $Q(x)$ and $R(x)$ with integer coefficients such that

$$x^a + x^b + x^c + x^d + x^e = Q(x)(x^5 + x^4 + x^2 + x + 1) + 2R(x).$$

Proposed by Michael Ren.

Answer. $\boxed{5208}$.

Solution. Work in $\mathbb{F}_2$. First, we claim that $x^5 + x^4 + x^2 + x + 1$ is irreducible. If it were reducible, it would have to be divisible by an irreducible polynomial of degree 1 or 2. Thus, we only have to consider divisibility by x, $x + 1$, and $x^2 + x + 1$. But $x^5 + x^4 + x^2 + x + 1 = x(x + 1)^2(x^2 + x + 1) + 1$, so it is not divisible by any of those, as desired.

Now consider a root z of $x^5+x^4+x^2+x+1$. Since $x^5+x^4+x^2+x+1$ is irreducible, z is an element of $\mathbb{F}_{32}$. Note that the order of a nonzero element of $\mathbb{F}_{32}$ divides 31, so it is either 1 or 31. Since the only element with order 1 is 1, z must have order 31, which means that it is a primitive root in $\mathbb{F}_{32}$. Hence, $1, z, z^2, \dots, z^{30}$ are the nonzero elements of $\mathbb{F}_{32}$. Furthermore, $x^5+x^4+x^2+x+1 \mid x^a+x^b+x^c+x^d+x^e$ if and only if $z^a + z^b + z^c + z^d + z^e = 0$. Hence, we just want to find the number of ways 5 distinct elements of $\mathbb{F}_{32}$ can add to 0.

We will first suppose that they are ordered and divide by 5! at the end. Note that we can just choose 4 random distinct elements and the fifth will be uniquely determined (it is actually their sum). This results in $31 \cdot 30 \cdot 29 \cdot 28$ ways. The only catch is that the fifth element might be the same as one of the first four or 0. To resolve this, we first count the number of ways 3 distinct elements of $\mathbb{F}_{32}$ can add to 0. By choosing 2 random distinct elements and taking their sum as the third, we have that there are $31 \cdot 30$ ways. It is not possible for their sum to be equal to one of them, because that implies that the other is 0. Now, the number of overcounted quintuples is simply $31 \cdot 30 \cdot 4 \cdot 28$, since there are 4 ways to choose three of the first four entries to sum to 0 and 28 ways to choose the element for the remaining entry and the last entry. Also, note that the number of ways for four elements to sum to 0 is $31 \cdot 30 \cdot 29 - 31 \cdot 30$ by a similar argument to what we had before. Hence, our answer is $\frac{31\cdot30\cdot29\cdot28-31\cdot30\cdot4\cdot28-31\cdot30\cdot29+31\cdot30}{5!} = \frac{31\cdot30\cdot24\cdot28}{120} = 5208.$ $\square$

24. For any positive integer n, let S_n denote the set of positive integers which cannot be written in the form $an + 2017b$ for nonnegative integers a and b. Let A_n denote the average of the elements of S_n if the cardinality of S_n is positive and finite, and 0 otherwise. Compute

$$\left\lfloor \sum_{n=1}^{\infty} \frac{A_n}{2^n} \right\rfloor.$$

Proposed by Tristan Shin.

Answer. $\boxed{840}$.

Solution. Let $m = 2017$. It is clear that $A_n > 0$ if and only if $(m, n) = 1$ and $n \ge 2$. Now, fix n that works.

I first claim that an integer k is not in S_n if and only if we can express $k = xm + yn$ with $x, y \in \mathbb{N}_0$ and $x \le n - 1$. The if direction is trivial. Assume that the only if direction is false. Take $k = x_0 m + y_0 n$ with $x_0 \ge n$ and let $z = \left\lfloor \frac{x_0}{n} \right\rfloor$. We then have that $k = x_0 m + y_0 n = (x_0 - nz)\, m + (y_0 + mz)\, n$. It is obvious that $x_0 - nz \in \mathbb{N}_0$, so this is a contradiction (take $x = x_0 - nz$ and $y = y_0 + mz$). Thus, the claim is true.

Now, what does this mean? This means that in the expression

$$\left(1 + x^m + x^{2m} + \ldots + x^{(n-1)m}\right)\left(1 + x^n + x^{2n} + \ldots\right) = \frac{1 - x^{mn}}{1 - x^m} \cdot \frac{1}{1 - x^n},$$

the generating function for the set of k with $k = xm + yn$ for $x, y \in \mathbb{N}_0$ and $x \leq n-1$, is also the generating function for the integers that are in S_n. Thus,

$$\frac{1}{1-x} - \frac{1-x^{mn}}{(1-x^m)(1-x^n)}$$

is the generating function for the integers that are in S_n. (Side note: combining these terms into one fraction and comparing the degrees of the numerator and denominator gives us the so-called Chicken McNugget Theorem.)

Thus, if

$$F(x) = \frac{1}{1-x} - \frac{1-x^{mn}}{(1-x^m)(1-x^n)},$$

then $|S_n| = \lim_{x\to 1} F(x)$ and the sum of the elements of S_n is $\lim_{x\to 1} F'(x)$.

We have that

$$F(x) = \frac{1}{1-x} - \frac{\sum_{i=0}^{n-1} x^{mi}}{1-x^n} = \frac{1 - x^n - \sum_{i=0}^{n-1} x^{mi} + \sum_{i=0}^{n-1} x^{mi+1}}{1 - x - x^n + x^{n+1}}.$$

To find the limit of F as $x \to 1$, we apply L'Hopital's rule twice:

$$\begin{aligned}
\lim_{x\to 1} F(x) &= \lim_{x\to 1} \frac{-n(n-1)x^{n-2} - \sum_{i=1}^{n-1} mi(mi-1)x^{mi-2} + \sum_{i=1}^{n-1} mi(mi+1)x^{mi-1}}{-n(n-1)x^{n-2} + (n+1)nx^{n-1}} \\
&= \frac{1}{2n}\left(-n(n-1) + 2m\sum_{i=1}^{n-1} i\right) = \frac{1}{2n}\left(-n(n-1) + mn(n-1)\right) \\
&= \frac{(m-1)(n-1)}{2}.
\end{aligned}$$

Therefore, the size of S_n is $|S_n| = \frac{(m-1)(n-1)}{2}$.

Now, rewrite F as

$$F(x) = \frac{1}{1-x} + \frac{\sum_{i=0}^{n-1} x^{mi}}{x^n - 1}.$$

Then

$$\begin{aligned}
F'(x) &= \frac{1}{(x-1)^2} + \frac{\left(\sum_{i=1}^{n-1} mix^{mi-1}\right)(x^n-1) - n\left(\sum_{i=0}^{n-1} x^{mi}\right)x^{n-1}}{(x^n-1)^2} \\
&= \frac{(x^n-1)^2 + \left(\sum_{i=1}^{n-1} mix^{mi-1}\right)(x^n-1)(x-1)^2 - n\left(\sum_{i=0}^{n-1} x^{mi}\right)x^{n-1}(x-1)^2}{(x-1)^2(x^n-1)^2} \\
&= \frac{\left(\sum_{i=0}^{n-1} x^i\right)^2 + \left(\sum_{i=1}^{n-1} mix^{mi-1}\right)(x^n-1) - n\left(\sum_{i=0}^{n-1} x^{mi}\right)x^{n-1}}{(x^n-1)^2} \\
&= \frac{\left(\sum_{i=0}^{n-1} x^i\right)^2 + \sum_{i=1}^{n-1}(mi-n)x^{mi+n-1} - \sum_{i=1}^{n-1} mix^{mi-1} - nx^{n-1}}{x^{2n} - 2x^n + 1}.
\end{aligned}$$

It follows from applying L'Hopital's rule twice that

$$\lim_{x\to 1} F'(x) = \lim_{x\to 1} \frac{\begin{gathered} 2\left(\sum_{i=1}^{n-1} ix^{i-1}\right)^2 + 2\left(\sum_{i=0}^{n-1} x^i\right)\left(\sum_{i=2}^{n-1} i(i-1)x^{i-2}\right) + \sum_{i=1}^{n-1}(mi-n)(mi+n-1)(mi+n-2)x^{mi+n-3} \\ - \sum_{i=1}^{n-1} mi(mi-1)(mi-2)x^{mi-3} - n(n-1)(n-2)x^{n-3} \end{gathered}}{2n(2n-1)x^{2n-2} - 2n(n-1)x^{n-2}}$$

$$= \frac{1}{2n^2}\left(\begin{gathered} \frac{n^2(n-1)^2}{2} + \frac{n^2(n-1)(2n-1)}{3} - n^2(n-1) - n(n-1)(n-2) \\ + \sum_{i=1}^{n-1}(mi-n)(mi+n-1)(mi+n-2) - mi(mi-1)(mi-2) \end{gathered}\right)$$

$$= \frac{(n-1)^2}{4} + \frac{(n-1)(2n-1)}{6} - \frac{n-1}{2} - \frac{(n-1)(n-2)}{2n} + \frac{1}{2n^2}\sum_{i=1}^{n-1} m^2ni^2 - mn^2i - n(n-1)(n-2)$$

$$= \begin{gathered} \frac{(n-1)^2}{4} + \frac{(n-1)(2n-1)}{6} - \frac{n-1}{2} - \frac{(n-1)(n-2)}{2n} + \frac{m^2(n-1)(2n-1)}{12} \\ - \frac{mn(n-1)}{4} - \frac{(n-1)^2(n-2)}{2n} \end{gathered}$$

$$= \frac{(m-1)(n-1)(2mn-m-n-1)}{12}$$

after mass simplification.

Thus, the sum of the elements of S_n is $\frac{(m-1)(n-1)(2mn-m-n-1)}{12}$, so the average of the elements of S_n is $\frac{2mn-m-n-1}{6}$.

Thus, if $y = \frac{1}{2}$,

$$\sum_{n=1}^{\infty} \frac{A_n}{2^n} = \sum_{n=1}^{\infty} \frac{(2m-1)n-(m+1)}{6}y^n - \frac{m-2}{6}y - \sum_{k=1}^{\infty} \frac{(2m-1)mk-(m+1)}{6}y^{mk}$$

$$= -\frac{m-2}{12} + \frac{2m-1}{6}\sum_{n=1}^{\infty} ny^n - \frac{m+1}{6}\sum_{n=1}^{\infty} y^n - \frac{m(2m-1)}{6}\sum_{k=1}^{\infty} k(y^m)^k + \frac{m+1}{6}\sum_{k=1}^{\infty}(y^m)^k$$

$$= -\frac{m-2}{12} + \frac{(2m-1)y}{6(1-y)^2} - \frac{(m+1)y}{6(1-y)} - \frac{m(2m-1)y^m}{6(1-y^m)^2} + \frac{(m+1)y^m}{6(1-y^m)}$$

$$= -\frac{m-2}{12} + \frac{2m-1}{3} - \frac{m+1}{6} - \frac{m(2m-1)2^m}{6(2^m-1)^2} + \frac{m+1}{6(2^m-1)}.$$

As

$$-\frac{m-2}{12} + \frac{2m-1}{3} - \frac{m+1}{6} = \frac{5m-4}{12} = 840 + \frac{1}{12}$$

and

$$-\frac{1}{12} < -\frac{2017 \cdot 4033 \cdot 2^{2017}}{6(2^{2017}-1)^2} + \frac{2018}{6(2^{2017}-1)} < \frac{11}{12}$$

is clear, we have that

$$\left\lfloor \sum_{n=1}^{\infty} \frac{A_n}{2^n} \right\rfloor = 840.$$

Note: Along the way, we divided out by certain terms at the right times to allow for easier applications of L'Hopital's rule. We take advantage of the fact that the number of times that L'Hopital's rule at a constant must be applied is the multiplicity of that constant in both the numerator and denominators. Straight-out using L'Hopital's rule would require 6 applications, but using the divisions shown in this solution only requires second derivatives. □

25. A *simple hyperplane* in $\mathbb{R}^4$ has the form

$$k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = 0$$

for some integers $k_1, k_2, k_3, k_4 \in \{-1, 0, 1\}$ that are not all zero. Find the number of regions that the set of all simple hyperplanes divide the unit ball $x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1$ into.

Proposed by Yannick Yao.

Answer. $\boxed{5376}$.

Solution. Note that it does not matter that we are looking at a unit ball since all simple hyperplane spass through the origin. We can replace it with $[-1, 1]^4$ or simply its surface and still have the same answer. This hypercube has 8 3-dimensional faces (that are separated by simple hyperplanes of the form $x_i - x_j = 0$ or $x_i + x_j = 0$ where $i \neq j$, and it suffices to consider one of the faces, say $x_4 = 1$. Within this face, the simple hyperplanes $x_1 = 0, x_2 = 0, x_3 = 0$ separate the face into $2^3 = 8$ identical unit cubes, and it suffices to consider one of them, say $0 \leq x_1, x_2, x_3 \leq 1$.

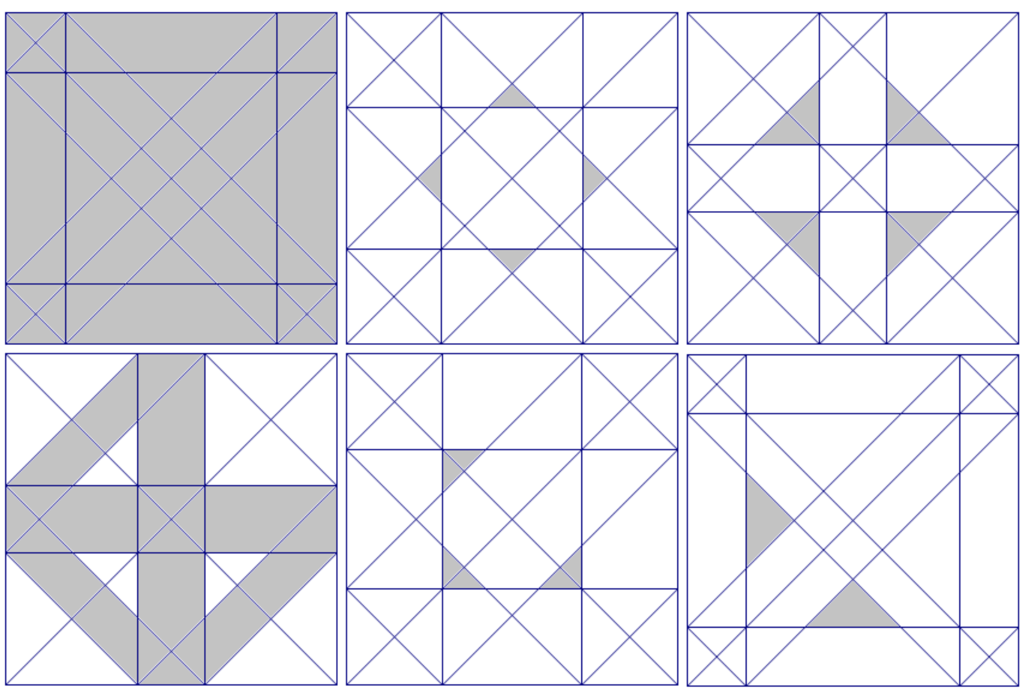
The simple hyperplanes that cut through this cube are:

$$x_1 = x_2, x_2 = x_3, x_3 = x_1, x_1 + x_2 = 1, x_1 + x_3 = 1, x_2 + x_3 = 1, x_1 + x_2 + x_3 = 1,$$

$$x_1+x_2-x_3 = 0, x_1+x_2-x_3 = 1, x_1-x_2+x_3 = 0, x_1-x_2+x_3 = 1, -x_1+x_2+x_3 = 0, -x_1+x_2+x_3 = 1.$$

(Note that the constant 1 is allowed since $x_4 = 1$ and one can set $k_4 = -1$.)

In order to visualize the arrangement of these cuts, one can consider their intersections with $x_3 = t$ for different values of t and draw them inside the unit square $0 \leq x_1, x_2 \leq 1$ as moving/stationary lines parametrized by t. The following six diagrams show the situation when $0 < t < \frac{1}{4}, \frac{1}{4} < t < \frac{1}{3}, \frac{1}{3} < t < \frac{1}{2}, \frac{1}{2} < t < \frac{2}{3}, \frac{2}{3} < t < \frac{3}{4}, \frac{3}{4} < t < 1$ respectively:



(Note that the asymmetry on the top-right corner comes from the absence of the plane $x_1+x_2+x_3=2$.)

Whenever three (or more) lines pass through a common point or when two parallel line coincide, a new region is created after they separate. These regions, along with the original ones, are shaded in gray in the above diagrams. One can count that there are $42+4+8+22+6+2=84$ regions in total in this unit cube. Therefore in total there are $8\cdot 8\cdot 84=5376$ regions in the original unit ball. $\square$

26. Let ABC be a triangle with $AB=13, BC=15, AC=14$, circumcenter O, and orthocenter H, and let M, N be the midpoints of minor and major arcs BC on the circumcircle of ABC. Suppose $P\in AB, Q\in AC$ satisfy that P, O, Q are collinear and $PQ||AN$, and point I satisfies $IP\perp AB, IQ\perp AC$. Let H' be the reflection of H over line PQ, and suppose $H'I$ meets PQ at a point T. If $\frac{MT}{NT}$ can be written in the form $\frac{\sqrt{m}}{n}$ for positive integers m, n where m is not divisible by the square of any prime, then find $100m+n$.

Proposed by Vincent Huang.

Answer. $\boxed{31418}$.

Solution. Let $S=PQ\cap BC$ and $S'\in BC$ with BQ, CP, AS' concurrent. Let $X=(APQ)\cap(ABC), X'=(SS'X)\cap(ABC)$, and A' be the antipode of A on (ABC).

It's clear that X is the center of a spiral similarity sending PQ to BC. By Ceva's Theorem on BQ, CP, AS' and $AP=AQ$ we know that $r=\frac{XP}{XQ}=\frac{BP}{CQ}=\frac{BS'}{CS'}$, so that XS' bisects $\angle BXC$, hence $X\in S'M$. Similarly, by Menelaus, $SB:SC=-r$, implying $X\in SN$. Since $X=(AI)\cap(AA')$ we deduce that $X\in IA'$. Now let $T'=XI\cap PQ$ so that $\angle PXQ$ is bisected by XT, implying that $\frac{PT}{TQ}=r$. Then the spiral similarity sending PQ to BC sends T' to S', so we deduce $XSS'T'$ is cyclic. But $MX\perp NX \implies SX\perp S'X$, so that T' is the projection of S' onto PQ.

Now I claim that $T=T'$. Since $(S, S'; B, C)$ is harmonic and $S'T\perp ST$ we deduce $S'T, ST$ bisect $\angle BTC$. It suffices to show that $T'H, T'I$, or equivalently $T'H, T'A'$, are isogonal in $\angle BT'C$, as this would imply that H', T', I are collinear.

To do this we note that $BPT'\sim CQT'$ by AA similarity, so that $\angle T'BA=\angle T'CA$. Then $\angle HBA=\angle HCA$ implies $\angle HBT'=\angle HCT'$. If we let $BH\cap CT'=B', CH\cap BT'=C'$ then $BCB'C'$ is cyclic, hence $T'B':T'C'=TB:TC$. Remark that $\angle BT'C$ and $\angle BAC$ have parallel angle bisectors. Then since $B'H'$ belongs to the direction $\perp_{AC}$, which is isogonal in $\angle BTC$ to the direction of BA', which is $\perp_{AB}$, and similarly for $C'H', CA'$, we can deduce that figures $T'B'HC', T'BA'C$ are inversely similar in $\angle BT'C$, hence $T'H, T'A'$ are isogonal as desired. (This paragraph can also be phrased in terms of hyperbolas, but I don't think it's necessary.)

Now we have established that $T=T'$. Note that (SS') must be an Apollonius circle in $\triangle XBC$, hence $XBX'C$ is harmonic so $X'B:X'C=r$, so the Angle Bisector Theorem yields that $X'\in SM, S'N$.

Since $(S, S'; B, C)$ is harmonic, we know $(SS'), (ABC)$ are orthogonal. Let the S-Apollonius circle of SMN meet (SS') at a point $T''\neq S$. Since (SS') and the S-Apollonius circle of SMN are both orthogonal to (ABC) we know the inversion about (ABC) swaps S, T'', so it follows that $T''=T$. Then $\frac{TM}{TN}=\frac{SM}{SN}=\frac{\sin SNM}{\sin SMN}=\frac{XM}{X'N}$.

We can verify that $XM=\frac{MB^2}{S'M}, NX'=\frac{NB^2}{NS'}$ so the ratio becomes $\frac{NS'}{S'M}\frac{MB^2}{NB^2}$.

Now we perform computations. $\cos A=\frac{5}{13}$ by Law of Cosines, $\sin 0.5A=\frac{2}{\sqrt{13}}, \cos 0.5A=\frac{3}{\sqrt{13}}$ by half-angle, $[ABC]=84$ by Heron's Formula, $R=\frac{65}{8}$ by $[ABC]=\frac{abc}{4R}$, $BM=2R\sin 0.5A=\frac{65}{2\sqrt{13}}, BN=2R\cos 0.5A=\frac{65\cdot 3}{4\sqrt{13}}$ by Law of Sines.

Next we can compute $AM=\frac{117}{2\sqrt{13}}$ by Ptolemy on $ABMC$. Then since $APMQ$ is a rhombus we know $AP=0.5AM\frac{1}{\cos 0.5A}=\frac{39}{4}$, so that $BP=\frac{13}{4}, CQ=\frac{17}{4}, r=\frac{13}{17}$.

Next, by this ratio we know $BS'=\frac{13}{2}, CS'=\frac{17}{2}$. By Stewart's Theorem on BMC we know $BM^2=MS'^2+BS'\cdot CS' \implies MS'=\sqrt{26}$. Similarly, we find $NS'=\frac{\sqrt{157\cdot 13}}{4}$.

Then $\frac{NS'}{S'M}\frac{MB^2}{NB^2} = \frac{\sqrt{314}}{18}$, so the answer is 31418. $\square$

27. Let N be the number of functions $f : \mathbb{Z}/16\mathbb{Z} \to \mathbb{Z}/16\mathbb{Z}$ such that for all $a, b \in \mathbb{Z}/16\mathbb{Z}$:

$$f(a)^2 + f(b)^2 + f(a+b)^2 \equiv 1 + 2f(a)f(b)f(a+b) \pmod{16}.$$

Find the remainder when N is divided by 2017.

Proposed by Zack Chroman.

Answer. $\boxed{793}$.

Solution. First, note that if we send $f(x) \to f(x) + 8$, the equation will still be true. Then WLOG assume that the image of f is contained within $\{0, \ldots, 7\}$, and we'll multiply by 2^{16} at the end to compensate.

Let $P(x, y)$ denote the statement $f(x)^2 + f(y)^2 + f(x+y)^2 \equiv 1 + 2f(x)f(y)f(x+y) \pmod{16}$. Note that $P(x, x) \implies 2f(x)^2 + f(2x)^2 \equiv 1 + 2f(x)^2 f(x+y) \pmod{16} \implies f(2x) \equiv 1 \pmod 2$.

So f maps evens to odds. Furthermore, for x,y odd, $P(x, y) \implies f(x)^2 + f(y)^2 \equiv 2f(x)f(y) \pmod 4 \implies (f(x) - f(y))^2 \equiv 0 \pmod 4$

So either f sends odds to odds or odds to evens.

Case 1: f sends odds to evens Then for x, y odd, $P(x, y) \implies f(x)^2 + f(y)^2 \equiv 0 \pmod 8$. Then $f(x) \equiv f(y) \pmod 4$.

Subcase 1: $f(\text{odd}) \equiv 0 \pmod 4$

Then $P(x, y)$ for x, y odd gives $f(x+y)^2 \equiv 1 \pmod{16}$, so $f(\text{even}) \in \{1, 5\}$. Now, $P(a, b)$ for a, b even gives $f(a)f(b)f(a+b) \equiv 1 \pmod 8$. That is, an even number of $f(a), f(b), f(a+b)$ are 5. In particular, $f(2a) = 1$.

If $a, b \equiv 2 \pmod 4$, $P(a, b) \implies f(a) = f(b)$. This value can be either 1 or 5, and by the above all equations will work out (checking P(odd,even) is straightforward). Then there are $2 \cdot 2^8$ possibilities in this case; 2 for all the 2 mod 4 numbers, and 2 for each odd, which can be 0 or 4.

Subcase 2: $f(\text{odd}) \equiv 2 \pmod 4$

Take x, y odd. Note that $f(x)^2 \equiv 4 \pmod{16}$.

$P(x, y) \implies 8 + f(x+y)^2 \equiv f(x)^2 + f(y)^2 + f(x+y)^2 \equiv 1 + 2f(x)f(y)f(x+y) \equiv 9 \pmod{16}$.

Then $f(\text{even}) \in \{1, 7\}$. Now for a, b even, $3 \equiv f(a)^2 + f(b)^2 + f(a+b)^2 \equiv 1 + 2f(a)f(b)f(a+b)$. Thus $f(a)f(b)f(a+b) \equiv 1 \pmod 8$. We finish as in subcase 1, and get that there are $2^8 \cdot 2$ possibilities in this subcase; 2 for assigning the evens, and 2 for each odd, which can be any of $\{2, 6\}$.

Case 2: f sends odds to odds Note that $P(x, y) \iff 3 \equiv 1 + 2f(x)f(y)f(x+y) \pmod 8$.

Thus $f(x)f(y)f(x+y) \equiv 1 \pmod 4$. In particular, $f(2x) \equiv 1 \pmod 4$. Going back to the equation, if x, y are of the same parity, $f(x) \equiv f(y) \pmod 4$.

Now, let $f(1) = a$. Then for any n, m,

$$\begin{aligned} P(n,m) &\iff f(n)^2 + f(m)^2 + f(n+m)^2 \equiv 1 + 2f(n)f(m)f(n+m) \pmod{16} \\ &\iff f(n)^2 + f(m)^2 + f(n+m)^2 + 8 \equiv 1 + 2f(n)f(m)f(n+m) + 8 \pmod{16} \\ &\iff f(n)^2 + (f(m)+4)^2 + f(n+m)^2 \equiv 1 + 2f(n)(f(m)+4)f(n+m) \pmod{16} \end{aligned}$$

This tells us that increasing or decreasing one of the values of f by 4 will not make a difference. We can then assume that $f(\text{odd}) = a$, and multiply by 2^7 at the end. Furthermore, since $f(2x) \equiv 1 \pmod 4$, assume $f(\text{even}) = 1$, and multiply by 2^8 at the end. It's easy to verify that any solution of this form works. There are 2^{17} solutions in this case; 2^8 for the even numbers and $4 \cdot 2^7$ for the odds.

The answer is $2^{16} \cdot (2^9 + 2^9 + 2^{17}) = 8657043456 \equiv 793 \pmod{2017}$ $\square$

28. Let S denote the set of fractions $\frac{m}{n}$ for relatively prime positive integers m and n with $m+n \le 10000$. The least fraction in S that is strictly greater than

$$\prod_{i=0}^{\infty}\left(1-\frac{1}{10^{2i+1}}\right)$$

can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $1000p+q$.

Proposed by James Lin.

Answer. $\boxed{3862291}$.

Solution. It's well known that $\dfrac{1}{\prod_{i=0}^{\infty}(1-x^{2i+1})} = \prod_{i=1}^{\infty}(1+x^i)$. Note that for $k \ge 1$, $[x^k]\prod_{i=1}^{\infty}(1+x^i)$ is the number of partitions of k into distinct positive integers. By splitting into the number of parts in each partition, one may compute the that

$$\prod_{i=1}^{\infty}(1+x^i) = 1+\sum_{i=1}^{\infty}\frac{x^{i(i+1)/2}}{(1-x)(1-x^2)\cdots(1-x^i)}.$$

In our scenario, for $x=\frac{1}{10}$, this means

$$\prod_{i=1}^{\infty}\left(1+\frac{1}{10^i}\right) = 1+\left(\frac{1}{9}+\frac{1}{9\cdot 99}\right)+\left(\frac{1}{9\cdot 99\cdot 999}+\frac{1}{9\cdot 99\cdot 999\cdot 9999}+\cdots\right) = 1+\frac{100}{891}+\epsilon,$$

where $\dfrac{1}{890109} < \epsilon < \dfrac{\frac{1}{890109}}{1-\frac{1}{9999}} < \dfrac{1}{890019}$.

Now, let F_n be the set of positive irreducible fractions less than 1 with denominator less than or equal to n. We need to find a lower bound $\frac{m}{n}$ of $N=\frac{100}{891}+\epsilon$ satisfying $m+2n \le 10000$ and $m \ge \frac{100}{891}n$, so we need to approximate N by a fraction that certainly is in F_{4735}. By the theory of Farey Sequences, the smallest element $\frac{a}{b}$ in F_{891} that is larger than $\frac{100}{891}$ must satisfy $891a-100b=1$ and $b \le 891$. We can solve this to get $a=11$ and $b=98$, and then note that $\frac{11}{98}-\frac{100}{891}=\frac{1}{98\cdot 891}=\frac{1}{87318}>\epsilon$. Then, by the theory of Farey Sequences, the fraction with the smallest denominator between $\frac{100}{891}$ and $\frac{11}{98}$ is $\frac{111}{989}$, and $\frac{111}{989}-\frac{100}{891}=\frac{1}{989\cdot 891}=\frac{1}{881199}>\epsilon$. The largest fraction in F_{4734} that is less than $\frac{111}{989}$ is $\frac{100+3\cdot 111}{891+3\cdot 989}=\frac{433}{3858}$. Note that $\frac{433}{3858}-\frac{100}{891}=\frac{1}{1145826}<\epsilon$. Hence, the desired fraction in S is $\frac{3858}{433+3858}=\frac{3858}{4291}$. Hence, our answer is 3862291.

Note: Our given expression is about 0.89909092, while $\frac{3858}{4291}\approx 0.89909112$ and $\frac{989}{1100}\approx 0.89909091$. □

29. Let ABC be a triangle with $AB=2\sqrt{6}, BC=5, CA=\sqrt{26}$, midpoint M of BC, circumcircle Ω, and orthocenter H. Let BH intersect AC at E and CH intersect AB at F. Let R be the midpoint of EF and let N be the midpoint of AH. Let AR intersect the circumcircle of AHM again at L. Let the

circumcircle of ANL intersect Ω and the circumcircle of BNC at J and O, respectively. Let circles AHM and JMO intersect again at U, and let AU intersect the circumcircle of AHC again at $V \neq A$. The square of the length of CV can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Find $100m + n$.

Proposed by Michael Ren.

Answer. $\boxed{1376029}$.

Solution. Consider Ψ, the inversion at A with power $AH * AD$. I claim that $\Psi(G) = S, \Psi(M) = Q$. This is pretty straightforward to verify, in particular noting that $\Psi(BC) = (AH)$. Then $\Psi(L) = AR \cap \Psi(H)\Psi(M) = AR \cap DQ$. Call this point L_a. Then since $\Psi(N)$ is the reflection of A over BC, call it N_a, $\Psi(J) = N_aL_a \cap EF$.

lemma : $\Psi(J)$ lies on AM.

This would imply that $J = AM \cap \Omega$.

proof : First, note that B, H, Q, C, N_a are cyclic on the reflection γ of Ω over BC. Then, $-1 = (D, S; B, C) \stackrel{H}{=} (N_a, Q; B, C)_\gamma = (A, Q*; B, C)_\Omega$.

Here $Q*$ is the reflection of Q over BC, which by the above lies on the $A-$symmedian. In particular, A, R, Q' collinear. Let $I = AR \cap BC \cap N_aQ$. Then, working over $\mathbb{RP}^2$,

$(A, Q; \Psi(J), M) \stackrel{N_a}{=} (D, I; N_aL_a \cap BC, M) \stackrel{L_a}{=} (D, A; N_a, ML_a \cap AD) \stackrel{M}{=} (I, A; Q', L_a) \stackrel{Q}{=} (N_a, A; \infty_{AD}, D) = -1$.

Therefore, $\Psi(J)$ lies on the polar of M with respect to circle $(AEHF)$, which is just line EF, as desired. This completes that lemma. As a corollary, note that J is the reflection of Q over M.

Lemma : O is the reflection of R over M.

Proof : By repeated power of a point, $MN * MO = AM * MJ = MB * MC = ME^2 = MR * MN$.

Lemma : J, M, O, S' are cyclic on the circle of diameter MS' **proof** : By the previous two lemmas, it suffices to show that S, R, Q, M, G are cyclic on the circle of diameter (SM). This follows from inversion with respect to the circle of diameter (BC), which sends $S \to D, R \to N, Q \to A, G \to H$.

Lemma : AG, AU are isogonal in $\angle BAC$

Proof : Let AU intersect (JMO) again at U_1, and let M_1 be the antipode of M on (AHM). Note that M', U, S' collinear. Then

$\angle S'MU_1 = \angle S'UU_1 = \angle AUM' = \angle AMM' = \angle DMH = \angle SMG$

Therefore, since G lies on (SM), we have $U_1 = G'$, the reflection of G over the perpendicular bisector of BC. Since $GG' \parallel BC$, and $G, G' \in \Omega$, it follows that lines $AG, AG' = AU$ are isogonal in $\angle BAC$, and in particular $BG = CG'$.

Lemma : $BG = CV$

Proof : Consider the circle at C with radius BG, call it ω. By the previous lemma $U_1 = G'$ lies on this circle. Clearly so does the reflection G'' of G' over AC. G'' lies on (AHC) since G' lies on (ABC).

Let P be the second intersection of $\omega, (AHC)$. By construction $\angle PAC = \angle CAG''' = \angle CAG'$, so $P \in AG'$. Therefore, $P = V$, and $V \in \omega$, as desired.

Since SBG and SAC are similar by antiparallels, $BG = \frac{BS \cdot AC}{AS}$. Let $AD = x$, $BD = y$, $CD = z$. We will find BS, AC, and AS in terms of x, y, and z.

Note that $DM \cdot DS = DB \cdot DC$, so $DS = \frac{2yz}{z-y}$. Then $BS = DS - y = \frac{y^2+yz}{z-y}$, and $AS^2 = DS^2 + AD^2 = x^2 + \frac{4y^2z^2}{(y-z)^2} = \frac{x^2(y-z)^2+4y^2z^2}{(y-z)^2}$. We also clearly have $AC^2 = x^2 + z^2$.

Thus, $BG^2 = \frac{(y^2+yz)^2(x^2+z^2)}{x^2(y-z)^2+4y^2z^2}$. Now it suffices to compute x, y, z. We have $y+z = 5$ and $y^2 - z^2 = -2$, so $y - z = -\frac{2}{5}$. This means that $y = \frac{23}{10}$ and $z = \frac{27}{10}$. The Pythagorean Theorem gives $x^2 = 24 - \frac{529}{100} = \frac{1871}{100}$.

Now, we just have to plug everything in for the answer, which is $\frac{23^2\cdot 50^2\cdot 26}{1871\cdot 4^2+4\cdot 23^2\cdot 27^2} = \frac{23^2\cdot 25^2\cdot 26}{1871\cdot 4+23^2\cdot 27^2} = \frac{23^2\cdot 25^2\cdot 26}{393125} = \frac{23^2\cdot 26}{629} = \frac{13754}{629}$. Therefore the answer is 1376029. □

30. Let $p = 2017$ be a prime. Given a positive integer n, let T be the set of all $n \times n$ matrices with entries in $\mathbb{Z}/p\mathbb{Z}$. A function $f : T \to \mathbb{Z}/p\mathbb{Z}$ is called an *n-determinant* if for every pair $1 \le i, j \le n$ with $i \neq j$,

$$f(A) = f(A'),$$

where A' is the matrix obtained by adding the jth row to the ith row.

Let a_n be the number of n-determinants. Over all $n \geq 1$, how many distinct remainders of a_n are possible when divided by $\dfrac{(p^p-1)(p^{p-1}-1)}{p-1}$?

Proposed by Ashwin Sah.

Answer. $\boxed{12106}$.

Solution. Notice that $\frac{(p^p-1)(p^{p-1}-1)}{p-1} = (p^{p-1}-1)\left(\frac{p^p-1}{p-1}\right)$, and that those two terms are relatively prime.

Let B, C be two rows of the matrix. Then an n-determinant remains constant when we transform them to $(B+C, C)$ and $(B+2C, C)$, and so on. Thus $(B+(p-1)C, C)$ is equivalent, so $(B-C, C)$ is as well and then $(B-C, B)$ is equivalent so $(-C, B)$ is as well after adding $p-1$ times. Thus we can "swap rows," to some extent.

Now if $xy \equiv 1 \pmod p$ then we can send (B, C) to $(B+xC, C)$ to $(B+xC, C-y(B+xC)) = (B+xC, -yB)$ to $(B+xC+x(-yB), -yB) = (xC, -yB)$ to (yB, xC). Thus we can "scale rows," to some extent.

Thus, we can easily perform steps equivalent to row reduction and reduce any matrix to a near-reduced row echelon form with the exact same n-determinant value.

If the rows are linearly dependent, we will obtain an all zero row and using it to scale other rows, we can reduce fully to reduced row echelon form.

Otherwise, we can reduce the matrix to the identity matrix, except the upper left element is the nonzero determinant of the original matrix, since the operations we perform preserve the determinant.

Furthermore, since the reduced row echelon form is unique, we can easily show that any matrix reduces to precisely one of these end matrices.

Thus the number of functions as desired is simply p^{S_n}, where S_n is the number of such final matrices.

Simple counting shows that $S_n = (p-1) + \sum_{k=0}^{n-1} p^{-\binom{k}{2}} s_k$, counting the nonzero determinant matrices with $p-1$ and counting the other rref forms with the latter sum, where $s_k = \sum_{a_1,\ldots,a_k \in \{0,1,\ldots,n-1\}} p^{a_1+\cdots+a_k}$ where we sum over distinct a_i. ($s_0 = 1$ since we must count the all zero matrix.)

Then $p^{S_n} \pmod{p^{p-1}-1}$ reduces to $p^{2^n-1} \pmod{p^{p-1}-1}$ while $p^{S_n} \pmod{\frac{p^p-1}{p-1}}$ reduces to $p^{n-1} \pmod{\frac{p^p-1}{p-1}}$.

Now the value of $n \pmod p$ determines the latter value - and all p possibilities are distinct -, while the value of $2^n \pmod{p-1}$ determines the former. There are p possibilities for the first. For the second, let $v = v_2(p-1)$ and $d = \text{ord}_{\frac{p-1}{2^v}}(2)$. Then $n = 1, \ldots, v-1$ give $v-1$ distinct values while $n = v, v+1, \ldots, v+d-1$ give d values which repeat forever.

This gives a total of $pd + (v-1)$ residues for the answer by the Chinese Remainder Theorem a bunch of times, since $\gcd(d, p) = 1$.

Now for $p = 2017$ we have $v = 5, d = 6$. Thus we find $2017 \cdot 6 + 4 = 12106$. □