The Online Math Open Spring Contest
Official Solutions
April 3 - 14, 2015

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1. What is the largest positive integer which is equal to the sum of its digits?

   Proposed by Evan Chen.

   Answer. $\boxed{9}$.

   Solution. All one-digit integers work, but for any integer n with more digits, the sum of the digits of n is strictly less than the value of n itself. Thus the answer is the largest one-digit integer, which is 9. $\square$

2. A classroom has 30 students, each of whom is either male or female. For every student S, we define his or her *ratio* to be the number of students of the opposite gender as S divided by the number of students of the same gender as S (including S). Let Σ denote the sum of the ratios of all 30 students. Find the number of possible values of Σ.

   Proposed by Evan Chen.

   Answer. $\boxed{2}$.

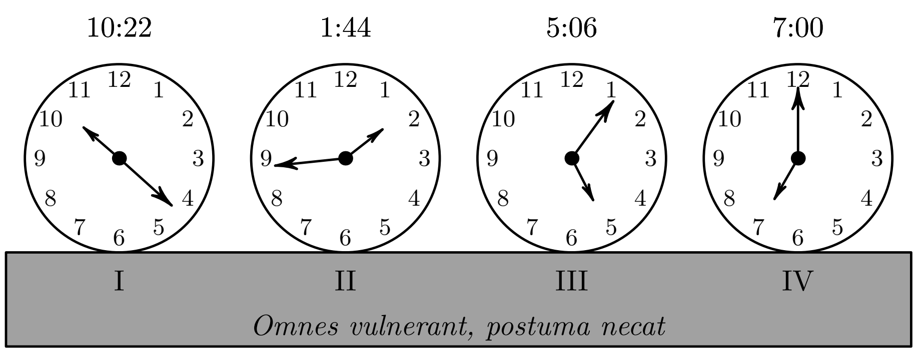
   Solution. We consider two cases. If there are $a \neq 0$ males and $b \neq 0$ females, then

$$\Sigma = \frac{b}{a} \cdot a + \frac{a}{b} \cdot b = a + b = 30.$$

   On the other hand, if all students are the same gender, then $\Sigma = 0$. Hence, there are two possible values of Σ. $\square$

   Remark. I apologize to all the teams who missed the edge case $\Sigma = 0$. I had noticed this when I wrote the problem, but did not think that it would actually cause any issues. Suffice to say in retrospect I would have explicitly mentioned it was possible that all students were the same gender.

3. On a large wooden block there are four twelve-hour analog clocks of varying accuracy. At 7PM on April 3, 2015, they all correctly displayed the time. The first clock is accurate, the second clock is two times as fast as the first clock, the third clock is three times as fast as the first clock, and the last clock doesn't move at all. How many hours must elapse (from 7PM) before the times displayed on the clocks coincide again? (The clocks do not distinguish between AM and PM.)



   Proposed by Evan Chen.

   Answer. $\boxed{12}$.

   Solution. It's clear that at least 12 hours must pass, since clock I accurately reflects the time while clock IV stands still at 7PM. But at twelve hours, clocks II and III also display the correct time of 7PM. Therefore, the answer is 12. $\square$

4. Find the sum of all distinct possible values of $x^2 - 4x + 100$, where x is an integer between 1 and 100, inclusive.

   Proposed by Robin Park.

   Answer. $\boxed{328053}$.

   Solution. Let $f(x) = x^2-4x+100$. Note that $f(1) = f(3)$ (because $x^2-4x+100 = (x-1)(x-3)+97$), so our answer is

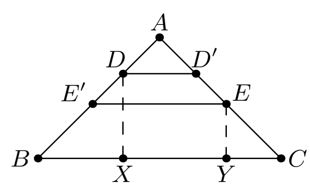
$$\sum_{k=1}^{100}(k^2 - 4k + 100) - f(1) = \frac{100 \cdot 101 \cdot 201}{6} - 4 \cdot \frac{100 \cdot 101}{2} + 100 \cdot 100 - 97 = 328053. \qquad \square$$

5. Let ABC be an isosceles triangle with $\angle A = 90^\circ$. Points D and E are selected on sides AB and AC, and points X and Y are the feet of the altitudes from D and E to side BC. Given that $AD = 48\sqrt{2}$ and $AE = 52\sqrt{2}$, compute XY.

   Proposed by Evan Chen.

   Answer. $\boxed{100}$.

   Solution. Let D' and E' be situated on AC and AB so that lines DD', EE', BC are parallel.



   From the diagram we see that XY is the average of DD' and EE' (why?). So the answer is

$$\frac{1}{2}(DD' + EE') = \frac{1}{2}(48 \cdot 2 + 52 \cdot 2) = 48 + 52 = 100.$$

$\square$

6. We delete the four corners of a 8×8 chessboard. How many ways are there to place eight non-attacking rooks on the remaining squares?

   Proposed by Evan Chen.

   Answer. $\boxed{21600}$.

   Solution. There are $6 \cdot 5$ ways to select a rook for the pair of farthest edges, and $6 \cdot 5$ ways for the remaining columns. Now it's straightforward to see that there are $4!$ ways to finish from here, since the central 6×6 square is essentially cut down to a 4×4 subsquares. Hence the answer is

$$(6 \cdot 5)^2 \cdot 4! = 21600. \qquad \square$$

7. A geometric progression of positive integers has n terms; the first term is 10^{2015} and the last term is an odd positive integer. How many possible values of n are there?

   Proposed by Evan Chen.

   Answer. $\boxed{8}$.

Solution. The point is to count the powers of two. Let $x_1 = 10^{2015}$, ..., $x_n =$ odd be the terms of the geometric progression. Call a_k the exponent of2 in the prime factorization of x_k. Then $a_1 = 2015$, $a_n = 0$, and the a_k form an *arithmetic* progression. If d is the common difference then we have $(n-1)d = 2015$. So the number of possible n correspond to the factors of 2015, of which there are 8.0 ☐

8. Determine the number of sequences of positive integers $1 = x_0 < x_1 < \cdots < x_{10} = 10^5$ with the property that for each $m = 0, \ldots, 9$ the number $\frac{x_{m+1}}{x_m}$ is a prime number.

   Proposed by Evan Chen.

   Answer. $\boxed{252}$.

   Solution. Since 10^5 has exactly 10 prime factors, we find that $\frac{x_{m+1}}{x_m} \in \{2, 5\}$ for each m. We must pick each of the two primes exactly 5 times. there are 10 positions, so the answer is $\binom{10}{5} = 252$. ☐

   Remark. This is equivalent to counting the number of lattice paths from $(0, 0)$ to $(5, 5)$.

9. Find the sum of the decimal digits of the number

$$5\sum_{k=1}^{99} k(k+1)(k^2+k+1).$$

   Proposed by Robin Park.

   Answer. $\boxed{72}$.

   Solution. We compute

$$\begin{aligned} 5\sum_{k=1}^{99}(k^2+k)(k^2+k+1) &= \sum_{k=1}^{99}(5k^4+10k^3+10k^2+5k) \\ &= \sum_{k=1}^{99}((k+1)^5-k^5) - \sum_{k=1}^{99} 1 \\ &= 100^5 - 1^5 - 99 = 9999999900 \end{aligned}$$

   Hence the answer is $9 \cdot 8 = 72$. ☐

10. Nicky has a circle. To make his circle look more interesting, he draws a regular 15-gon, 21-gon, and 35-gon such that all vertices of all three polygons lie on the circle. Let n be the number of distinct vertices on the circle. Find the sum of the possible values of n.

    Proposed by Yang Liu.

    Answer. $\boxed{326}$.

    Solution. Note that if a regular m-gon and a regular n-gon intersect at least one point, then they intersect at exactly $\gcd(m, n)$ points. Now we claim that if two pairs of polygons intersect, then all three intersect at some unique point. Indeed, in this case we can split up the initial circle evenly into 105 points, for which every 3 points lie on the 35-gon, every 5 points lie on the 21-gon, and every 7 points lie on the 15-gon, with variable starting positions. So one of these 105 points is a common vertex of all three polygons if it is the solution $x \pmod{105}$ to some set of congruences

$$\begin{aligned} x &\equiv a \pmod 3 \\ x &\equiv b \pmod 5 \\ x &\equiv c \pmod 7 \end{aligned}$$

But by the Chinese Remainder Theorem, there is exactly one unique solution x to this set of congruences, so the claim is proven.

Now we split the problem into several cases based on intersecting pairs.

If no two polygons have a common vertex, there are $15 + 21 + 35 = 71$ distinct vertices on the circle.

If the 15-gon and 21-gon share 3 vertices but neither shares a vertex with the 35-gon, there are 68 distinct vertices. Similarly, in the cases for the other two pairs intersecting we get $71 - 5 = 66$ and $71 - 7 = 64$ vertices.

If all three polygons intersect, by the Principle of Inclusion-Exclusion there are a total of $15 + 21 + 35 - 3 - 5 - 7 + 1 = 57$ distinct vertices. Thus the final answer is $71 + 68 + 66 + 64 + 57 = 326$. $\square$

11. Let S be a set. We say S is *D^*-finite* if there exists a function $f : S \to S$ such that for every nonempty proper subset $Y \subsetneq S$, there exists a $y \in Y$ such that $f(y) \notin Y$. The function f is called a *witness* of S. How many witnesses does $\{0, 1, \cdots, 5\}$ have?

    Proposed by Evan Chen.

    Answer. $\boxed{120}$.

    Solution. Clearly any witness f must in fact be surjective; otherwise we can take Y to be the range of f. Taking the cycle decomposition of f, we find that f must consist of only a single cycle, or else we could take Y to be any such cycle. However, we can also see that any such "single cycle" works.

    Then, by a standard argument, there are $5! = 120$ such witnesses. $\square$

    Remark. D^*-finiteness is equivalent to finiteness in the usual sense, and gives a characterization of "finite set" in ZFC.

12. At the Intergalactic Math Olympiad held in the year 9001, there are 6 problems, and on each problem you can earn an integer score from 0 to 7. The contestant's score is the *product* of the scores on the 6 problems, and ties are broken by the sum of the 6 problems. If 2 contestants are still tied after this, their ranks are equal. In this olympiad, there are $8^6 = 262144$ participants, and no two get the same score on every problem. Find the score of the participant whose rank was $7^6 = 117649$.

    Proposed by Yang Liu.

    Answer. $\boxed{1}$.

    Solution. Since there exactly 7^6 ways to obtain a nonzero total score, we discover everyone underneath the 117649th rank scored a zero. Hence this contestant's score must be $1^6 = 1$, the lowest possible nonzero score. $\square$

13. Let ABC be a scalene triangle whose side lengths are positive integers. It is called *stable* if its three side lengths are multiples of 5, 80, and 112, respectively. What is the smallest possible side length that can appear in any stable triangle?

    Proposed by Evan Chen.

    Answer. $\boxed{20}$.

    Solution. Suppose that the three side lengths are $5a$,$80b$, $112c$, where a, b, c are positive integers. By the triangle inequality, we must have

    $$5a > |80b - 112c| = 16|5b - 7c|$$

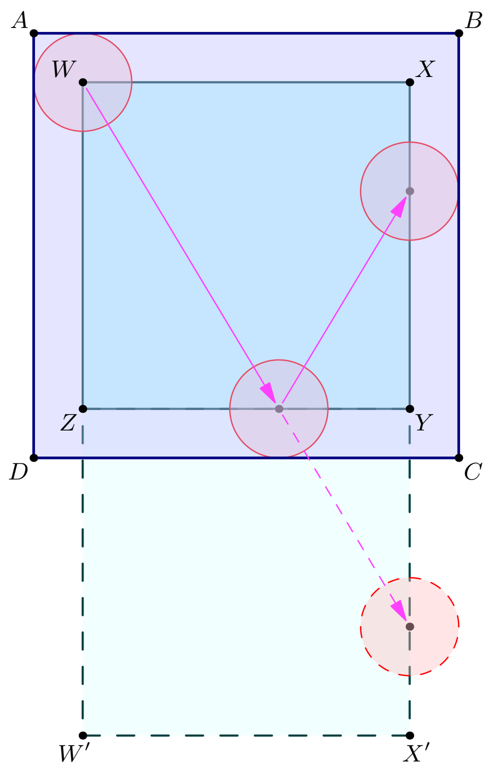
    noting that the right hand side cannot be 0, as the triangle is scalene. So $5a \geq 16$, and thus $5a \geq 20$. Clearly $80b$ and $112c$ are at least 20, so the smallest side length is always at least 20. Conversely, we choose b, c such that $|5b - 7c| = 1$ holds (which is possible since $5, 7$ are relatively prime): $b = 3$ and $c = 2$ suffice. Therefore the triangle with side lengths $20, 240$, and 224, which is a triangle, gives 20 as the minimum possible side length. $\square$

14. Let $ABCD$ be a square with side length 2015. A disk with unit radius is packed neatly inside corner A (i.e. tangent to both $\overline{AB}$ and $\overline{AD}$). Alice kicks the disk, which bounces off $\overline{CD}$, $\overline{BC}$, $\overline{AB}$, $\overline{DA}$, $\overline{DC}$ in that order, before landing neatly into corner B. What is the total distance the center of the disk travelled?

    Proposed by Evan Chen.

    Answer. $\boxed{10065}$.

    Solution. The main idea is to treat the disk as a *single point*, namely its center. If we consider an inner 2013×2013 square then we can treat the center of the ball as a particle being reflected by the fake "edges" of this inner square. See the diagram:



    With the insight, upon drawing out the various bounces, we see that the ball travels the diagonal of a $3M \times 4M$ rectangle, where $M = 2013$. This is $5M = 10065$. $\square$

15. Let a, b, c, and d be positive real numbers such that

$$a^2 + b^2 - c^2 - d^2 = 0 \quad \text{and} \quad a^2 - b^2 - c^2 + d^2 = \frac{56}{53}(bc + ad).$$

    Let M be the maximum possible value of $\frac{ab+cd}{bc+ad}$. If M can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers, find $100m + n$.

    Proposed by Robin Park.

    Answer. $\boxed{4553}$.

    Solution. Consider a quadrilateral $ABCD$ with sidelengths $AB = a$, $BC = b$, $CD = c$, $DA = d$, and $\angle B = \angle D = 90^\circ$ (which is possible by the first equation). By the second equation and the Law of Cosines, we have that $\cos A = -\cos C = \frac{28}{53}$. Now we compute the area of $ABCD$ in two ways by

$$[ABCD] = \frac{1}{2}(ab + cd) = \frac{1}{2}(bc + da)\sin A = \frac{45}{106}(bc + ad)$$

so $\frac{ab+cd}{bc+ad} = \frac{45}{53}$. $\square$

16. Joe is given a permutation $p = (a_1, a_2, a_3, a_4, a_5)$ of $(1, 2, 3, 4, 5)$. A *swap* is an ordered pair (i, j) with $1 \le i < j \le 5$, and this allows Joe to swap the positions i and j in the permutation. For example, if Joe starts with the permutation $(1, 2, 3, 4, 5)$, and uses the swaps $(1, 2)$ and $(1, 3)$, the permutation becomes

$$(1, 2, 3, 4, 5) \to (2, 1, 3, 4, 5) \to (3, 1, 2, 4, 5).$$

Out of all $\binom{5}{2} = 10$ swaps, Joe chooses 4 of them to be in a set of swaps S. Joe notices that from p he could reach any permutation of $(1, 2, 3, 4, 5)$ using only the swaps in S. How many different sets are possible?

Proposed by Yang Liu.

Answer. $\boxed{125}$.

Solution. Consider the graph on 5 vertices with vertices numbered 1 to 5. For each swap (i, j) draw an edge between i and j. The key observation is that all other permutations are reachable from the starting one if and only if the graph is connected (why?).

Since we chose exactly 4 edges, it is connected if and only if it is a tree. So the problem amounts to counting the number of *spanning trees* on K_5. A famous theorem of Cayley says that in general, the number of spanning trees of K_n is $n^{n-2} = 5^3 = 125$. (However, for the small value $n = 5$ one can also enumerate the trees by hand.) $\square$

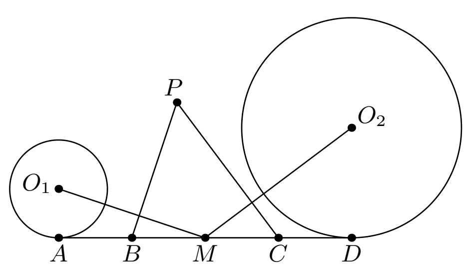
17. Let A, B, M, C, D be distinct points on a line such that $AB = BM = MC = CD = 6$. Circles ω_1 and ω_2 with centers O_1 and O_2 and radius 4 and 9 are tangent to line AD at A and D respectively such that O_1, O_2 lie on the same side of line AD. Let P be the point such that $PB \perp O_1M$ and $PC \perp O_2M$. Determine the value of $PO_2^2 - PO_1^2$.

Proposed by Ray Li.

Answer. $\boxed{65}$.

Solution. The length of $AB = BM = MC = CD$ is extraneous, and can be weakened to just $AB = BM$, $MC = CD$.

The key observation is that P is the radical center of ω_1, ω_2 and the circle of radius zero centered at M. Indeed PB is the radical axiom of ω_1 and M, since it passes through the point B, and is perpendicular to the line O_1M through the centers. Similarly PC is a radical axis.



Consequently, $PO_1^2 - 4^2 = PO_2^2 - 9^2$, by the characterization of the power of a point through distances. Thus $PO_2^2 - PO_1^2 = 9^2 - 4^2 = 65$. $\square$

Remark. In fact, the radical axis of ω_1 and ω_2 is exactly line PM.

18. Alex starts with a rooted tree with one vertex (the root). For a vertex v, let the size of the subtree of v be $S(v)$. Alex plays a game that lasts nine turns. At each turn, he randomly selects a vertex in the tree, and adds a child vertex to that vertex. After nine turns, he has ten total vertices. Alex selects one of these vertices at random (call the vertex v_1). The expected value of $S(v_1)$ is of the form $\frac{m}{n}$ for relatively prime positive integers m, n. Find $m + n$.

    Note: In a rooted tree, the subtree of v consists of its indirect or direct descendants (including v itself).

    Proposed by Yang Liu.

    Answer. $\boxed{9901}$.

    Solution. The answer is $H_{10} = \frac{1}{1} + \cdots + \frac{1}{10}$. We show that in general, for n vertices (rather than 10) there answer is H_n

    We show that given any tree on n nodes, the expected of $S(v)$ increases by $\frac{1}{n+1}$ when we add a new node. For a node v, define $d(v)$ be the number of nodes on the path from v to the root. By a simple double count,

$$\sum_v d(v) = \sum_v S(v).$$

    Clearly, when we randomly add a new child, we expect to increase $\sum_v d(v)$ by $\frac{1}{n} \cdot \sum_v d(v) + 1$.

    So our new average is

$$\frac{1}{n+1} \cdot \left(\sum d(v) + \frac{1}{n} \cdot \sum d(v) + 1 \right) = \frac{1}{n} \cdot \sum d(v) + \frac{1}{n+1}.$$

    Therefore, our answer is H_n, as claimed.

    It is now direct to compute $H_{10} = \frac{7381}{2520}$, for an answer of 9901. $\square$

19. Let ABC be a triangle with $AB = 80, BC = 100, AC = 60$. Let D, E, F lie on BC, AC, AB such that $CD = 10, AE = 45, BF = 60$. Let P be a point in the plane of triangle ABC. The minimum possible value of $AP + BP + CP + DP + EP + FP$ can be expressed in the form $\sqrt{x} + \sqrt{y} + \sqrt{z}$ for integers x, y, z. Find $x + y + z$.

    Proposed by Yang Liu.

    Answer. $\boxed{15405}$.

    Solution. The main observation is that AD, BE, and CF concur; since we have $AP + PD \leq AD$ with equality if P lies on AD, and similarly for the other cevians, we find that

$$AP + BP + CP + DP + EP + FP \leq AD + BE + CF$$

    and that equality occurs if P is the concurrency point. By the Pythagorean Theorem, we have

$$BE^2 = 80^2 + 45^2 \quad \text{and} \quad CF^2 = 60^2 + 20^2.$$

    Moreover, by the Law of Cosines on $\triangle ADC$ (or by Stewart's Theorem or several other means), we find

$$AD^2 = 60^2 + 10^2 - 2 \cdot 60 \cdot 10 \cdot \frac{60}{100} = 60^2 + 10^2 - 720.$$

    Hence the quantity requested in the problem is

$$10^2 + 20^2 + 45^2 + 2 \cdot 60^2 + 80^2 - 720 = 15405.$$

    $\square$

20. Consider polynomials P of degree 2015, all of whose coefficients are in the set $\{0, 1, \ldots, 2010\}$. Call such a polynomial *good* if for every integer m, one of the numbers $P(m) - 20$, $P(m) - 15$, $P(m) - 1234$ is divisible by 2011, and there exist integers m_{20}, m_{15}, m_{1234} such that $P(m_{20}) - 20, P(m_{15}) - 15, P(m_{1234}) - 1234$ are all multiples of 2011. Let N be the number of good polynomials. Find the remainder when N is divided by 1000.

   Proposed by Yang Liu.

   Answer. $\boxed{460}$.

   Solution. Note that 2011 is a prime. The following claim is the most important part of our solution:

   Lemma 20.1. *Given the values* $P(0), P(1), \ldots, P(2010)$, *there exists a unique polynomial of degree at most* 2010 *and coefficients in the set* $\{0, 1, \ldots, 2010\}$ *such that* $P(x) \equiv x \pmod{2011}$ *for all* x.

   Proof. Work in $\mathbb{F}_{2011}[x]$ throughout this proof. This is essentially polynomials with coefficients as integers $\pmod{2011}$. First I will prove uniqueness. If $P(x), Q(x)$ are the same $\pmod{2011}$ for all x, then $x^{2011} - x | P(x) - Q(x)$, contradicting the fact that $\max(\deg P, \deg Q) \le 2010$.

   Now I show existence. By the Lagrange Interpolation formula,

$$P(x) = \sum_{i=0}^{2010} P(i) \frac{\prod_{0 \le j \le 2010, j \ne i}(x - j)}{\prod_{0 \le j \le 2010, j \ne i}(i - j)}.$$

   Since every integer has a unique inverse $\pmod{2011}$ and no denominators are a multiple of 2011, existence is shown. $\square$

   Now we will count the number of ways to assign the values $20, 15, 1234$ to $P(0), P(1), \ldots, P(2010)$ such that all 3 values are used at least once. This is essentially *Stirling Numbers of the Second Kind* and can be computed with Principle of Inclusion-Exclusion. The answer is $3^{2011} - 3 \cdot 2^{2011} + 3 \cdot 1^{2011}$.

   Finally, since $P(x)$ is degree 2015, write $P(x) = (x^{2011} - x) \cdot Q(x) + R(x)$, where $\deg R(x) < 2011, \deg Q(x) = 4$. So given the values of $P(0), P(1), \ldots, P(2010)$, by the above claim, $R(x)$ is determined uniquely. $(2011 | x^{2011} - x$ for all $x)$ $Q(x)$ can be chosen in $2010 \cdot 2011^4$ ways, just by choosing the coefficients independently.

   Therefore, our final answer is $2010 \cdot 2011^4 \cdot (3^{2011} - 3 \cdot 2^{2011} + 3 \cdot 1^{2011}) \equiv 460 \pmod{2011}$. $\square$

21. Let $A_1A_2A_3A_4A_5$ be a regular pentagon inscribed in a circle with area $\frac{5+\sqrt{5}}{10}\pi$. For each $i = 1, 2, \ldots, 5$, points B_i and C_i lie on ray $\overrightarrow{A_iA_{i+1}}$ such that

$$B_iA_i \cdot B_iA_{i+1} = B_iA_{i+2} \quad \text{and} \quad C_iA_i \cdot C_iA_{i+1} = C_iA_{i+2}^2$$

   where indices are taken modulo 5. The value of $\frac{[B_1B_2B_3B_4B_5]}{[C_1C_2C_3C_4C_5]}$ (where $[\mathcal{P}]$ denotes the area of polygon $\mathcal{P}$) can be expressed as $\frac{a+b\sqrt{5}}{c}$, where a, b, and c are integers, and $c > 0$ is as small as possible. Find $100a + 10b + c$.

   Proposed by Robin Park.

   Answer. $\boxed{101}$.

   Solution. First, note that the area condition implies that the side length of the pentagon is 1. (This can be shown using trigonometry or similar) Let the circumcircle of the pentagon be ω. Next I will show that $C_iA_{i+2} = 1$. Because $C_iA_{i+2}^2 = C_iA_i \cdot C_iA_{i+1}$, C_i is the intersection of the tangent from A_{i+2} to ω with A_iA_{i+1}.

   Therefore, $\angle C_iA_{i+1}A_{i+2} = 180^\circ - \angle A_iA_{i+1}A_{i+2} = 72^\circ$. Also, $\angle C_iA_{i+2}A_{i+1} = \angle A_{i+2}A_iA_{i+1} = 36^\circ$ because of tangency. So $\angle A_{i+1}CA_{i+2} = 72^\circ = \angle C_iA_{i+1}A_{i+2} \implies CA_{i+2} = A_{i+1}A_{i+2} = 1$. This in fact means that $B_i \equiv C_i$ because $1^2 = 1$.

   Therefore the pentagons are equal, so the ratio is 1. Our final answer would be $\frac{1+0\sqrt{5}}{1} \implies 101$. $\square$

22. For a positive integer n let $n\#$ denote the product of all primes less than or equal to n (or 1 if there are no such primes), and let $F(n)$ denote the largest integer k for which $k\#$ divides n. Find the remainder when $F(1) + F(2) + F(3) + \cdots + F(2015\# - 1) + F(2015\#)$ is divided by 3999991.

    Proposed by Evan Chen.

    Answer. $\boxed{240430}$.

    Solution. The key observation is that by double-counting, the sum is equal to

$$\sum_{k\#|n} 1 = \sum_{1 \le k \le 2016} \left\lfloor \frac{N}{k\#} \right\rfloor$$

where $N = 2015\# = 2016\#$ (note $N < 2017\#$).

The quantity in the floor is an integer, and in fact it's the product of the primes up to 2011 which are strictly greater than k.

Note that $M = 3999991 = 2000^2 - 3^2 = 1997 \cdot 2003$, both of which are primes. So in this sum all terms $k < 1997$ vanish mod M. The primes greater than 1997 are 1999, 2003, 2011 and 2017.

- For $1997 \le k \le 1998$ we have $1999 \cdot 2003 \cdot 2011$, which contributes $2(2 \cdot 2003 \cdot 14) \equiv 56 \cdot 2003 \pmod{M}$.
- For $1999 \le k \le 2002$ we have $2003 \cdot 2011$, which contributes $4(2003 \cdot 14) \equiv 56 \cdot 2003 \pmod{M}$.
- For $2003 \le k \le 2010$ we have 2011 which contributes $8 \cdot 2011$.
- For $2011 \le k \le 2016$ we have 1 which contributes 6.

Hence the answer is $112 \cdot 2003 + 8 \cdot 2011 + 6 = 224336 + 16088 + 6 = 240430$. □

23. Let $N = 12!$ and denote by X the set of positive divisors of N other than 1. An *pseudo-ultrafilter* U is a nonempty subset of X such that for any $a, b \in X$:

    - If a divides b and $a \in U$ then $b \in U$.
    - If $a, b \in U$ then $\gcd(a, b) \in U$.
    - If $a, b \notin U$ then $\operatorname{lcm}(a, b) \notin U$.

    How many such pseudo-ultrafilters are there?

    Proposed by Evan Chen.

    Answer. $\boxed{19}$.

    Solution. Suppose U is a pseudo-ultrafilter. Note that inductively, the greatest common divisor of any n elements of U must also be in U. Let g be the greatest common divisor of all elements of U; then $g \in U$ and $g > 1$. So all elements of U are multiples of g, and by the first condition, all multiples of g are elements of U. Thus we simply want to find all $g \in X$ such that

$$\{h \in x | g \mid h\}$$

is a pseudo-ultrafilter. This clearly satisfies the first two conditions, so it suffices to find g satisfying the third condition.

If there are some integers $a, b > 1$ with $g = ab$ and $\gcd(a, b) = 1$, then we have a contradiction, as $\operatorname{lcm}(a, b) = g$. Therefore any valid g must satisfy $g = p^k$ for some prime p and $k \ge 1$ (as g cannot be 1). Conversely, if $g = p^k$, then $p^k \nmid a$ and $p^k \nmid b$ implies $p^k \nmid \operatorname{lcm}(a, b)$. So it suffices to count the number of prime powers which are factors of 12!, which is $10 + 5 + 2 + 1 + 1 = 19$.

□

24. Suppose we have 10 balls and 10 colors. For each ball, we (independently) color it one of the 10 colors, then group the balls together by color at the end. If S is the expected value of the square of the number of distinct colors used on the balls, find the sum of the digits of S written as a decimal.

    Proposed by Michael Kural.

    Answer. $\boxed{55}$.

    Solution. We solve the problem for the general case of n balls and n colors for $n \geq 2$.

    Let C be the set of colors, so $|C| = 10$. Let B be the set of colors which are the color of some ball, so we want to find $E[|B|^2]$. For a set $T \subseteq C$, let $f(T)$ be the probability that $B = T$, and let $g(T)$ be the probability that $B \subseteq T$. So then

$$E[|B|^2] = \sum_{T \subseteq C} f(T)|T|^2$$

and also

$$g(T) = \sum_{R \subseteq T} f(R)$$

Clearly, $g(T) = \frac{|T|^n}{n^n}$. By Mobius Inversion,[1]

$$f(T) = \sum_{R \subseteq T} (-1)^{|T|-|R|} g(R)$$

Thus

$$E[|B|]^2 = \sum_{T \subseteq C} \sum_{R \subseteq T} (-1)^{|T|-|R|} g(R)|T|^2 = \sum_{R \subseteq C} \sum_{R \subseteq T \subseteq C} (-1)^{|T|-|R|} g(R)|T|^2$$
$$= \sum_{i=0}^{n} \sum_{j=i}^{n} \binom{n}{i} \binom{n-i}{j-i} (-1)^{j-i} \frac{i^n}{n^n} j^2$$

where we set $i = |R|, j = |T|$, since there are $\binom{n}{i}$ choices for R given fixed i and $\binom{n-i}{j-i}$ choices for T given j and $R \subseteq T$.

Lemma 24.1. *If x is a real number and p, q are nonnegative integers with $p > q$, then*

$$\sum_{k=0}^{p} (-1)^k \binom{p}{k} (x+k)^q = 0$$

Proof. This is simply saying that the pth finite difference of a qth degree polynomial is 0, which is true because each successive finite difference decreases the degree of the polynomial by 1. $\square$

Then note that if $n - i > 2$,

$$\sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{j-i} j^2 = 0.$$

so our expression for $E[|B|^2]$ simply reduces to

$$E[|B|^2] = \sum_{i=n-2}^{n} \sum_{j=i}^{n} \binom{n}{i} \binom{n-si}{j-i} (-1)^{j-i} \frac{i^n}{n^n} j^2.$$

Evaluating this sum yields

$$E[|B|^2] = \binom{n}{n-2} \frac{(n-2)^n}{n^n} \left(n^2 - 2(n-1)^2 + (n-2)^2\right)$$

[1] This is also essentially Principle of Inclusion-Exclusion; there are a lot of ways to see that this identity holds.

$$+\binom{n}{n-1}\frac{(n-1)^n}{n^n}\left((n-2)^2-(n-1)^2\right)+\binom{n}{n}\frac{n^n}{n^n}\left(n^2\right)$$

which simplifies to

$$\frac{1}{n^n}\left(n(n-1)(n-2)^n+n(1-2n)(n-1)^n+n^{n+2}\right)$$

For $n=10$, this is

$$43.414772797$$

yielding a final answer of 55. □

25. Let $V_0=\varnothing$ be the empty set and recursively define V_{n+1} to be the set of all $2^{|V_n|}$ subsets of V_n for each $n=0,1,\dots$. For example

$$V_2=\{\varnothing,\{\varnothing\}\} \quad \text{and} \quad V_3=\{\varnothing,\{\varnothing\},\{\{\varnothing\}\},\{\varnothing,\{\varnothing\}\}\}.$$

A set $x\in V_5$ is called *transitive* if each element of x is a subset of x. How many such transitive sets are there?

Proposed by Evan Chen.

Answer. $\boxed{4131}$.

Solution. For ease of notation, let $0=\varnothing$, $1=\{0\}$, $2=\{0,1\}$, $3=\{0,1,2\}$. Then

$$V_3=\{0,1,2,\{1\}\}.$$

Let $x\in V_5$, i.e. $x\subseteq V_4$, be transitive. We now count the number of transitive sets in V_4 by casework on $y=x\cap V_3$.

- If $y=\varnothing$, this means $x=\varnothing$, so one set here.
- If $y=\{0\}$, this means $x=1$, so one set here.
- If $y=\{0,1\}$, this means $x=2$, so one set here.
- If $y=\{0,1,2\}=3$, then there are four more elements of V_4 we can add to x (namely the $2^3-4=4$ subsets of 3 not already in V_2), so $2^4=16$ cases here.
- If $y=\{0,1,\{1\}\}$, we again get $2^4=16$ cases.
- If $y=V_3$, then we can add any of the $2^4-4=12$ elements of $V_4\setminus V_3$ we like, giving $2^{12}=4096$ cases.

Tallying the total gives $4096+32+3=4131$. □

Remark. The sets V_i in the problem are the levels of the **von Neumann universe**. The phrase "transitive" reflects that $\in$ is a transitive relation on the elements of x.

26. Consider a sequence $T_0,T_1,\dots$ of polynomials defined recursively by $T_0(x)=2$, $T_1(x)=x$, and $T_{n+2}(x)=xT_{n+1}(x)-T_n(x)$ for each nonnegative integer n. Let L_n be the sequence of Lucas Numbers, defined by $L_0=2$, $L_1=1$, and $L_{n+2}=L_n+L_{n+1}$ for every nonnegative integer n.

Find the remainder when $T_0(L_0)+T_1(L_2)+T_2(L_4)+\cdots+T_{359}(L_{718})$ is divided by 359.

Proposed by Yang Liu.

Answer. $\boxed{5}$.

Solution. Let $\alpha = \frac{3+\sqrt{5}}{2}$, so that $L_{2n} = \alpha^n + \alpha^{-n}$. Note both 359 and 179 are prime. Now since

$$\left(\frac{5}{359}\right) = 1$$

(i.e. 5 is a quadratic residue modulo 359, as can be checked by, say, quadratic reciprocity), we have that $\alpha \in \mathbb{F}_{359}$. Finally,

$$\alpha = \left(\frac{1+\sqrt{5}}{2}\right)^2,$$

so

$$\alpha^{179} = 1.$$

By properties of Chebyschev polynomials, $T_n(L_{2n}) = \alpha^{n^2} + \alpha^{-n^2}$.

To finish, note that since $179 \equiv 3 \pmod 4$, $-n^2$ is not a quadratic residue modulo 179. Therefore,

$$\sum_{n=0}^{178} T_n\left(L_{2n}\right) = \sum_{n=0}^{178}\left(\alpha^{n^2} + \alpha^{-n^2}\right) = 2(\alpha^0 + \alpha^1 + \ldots + \alpha^{178}) = 0$$

since we run through each residue and non-residue precisely twice.

So in our desired sum almost everything goes to 0, except the last two terms, which are easily computed to be $2 + 3 = 5$. $\square$

27. Let $ABCD$ be a quadrilateral satisfying $\angle BCD = \angle CDA$. Suppose rays AD and BC meet at E, and let Γ be the circumcircle of ABE. Let Γ_1 be a circle tangent to ray CD past D at W, segment AD at X, and internally tangent to Γ. Similarly, let Γ_2 be a circle tangent to ray DC past C at Y, segment BC at Z, and internally tangent to Γ. Let P be the intersection of WX and YZ, and suppose P lies on Γ. If F is the E-excenter of triangle ABE, and $AB = 544, AE = 2197, BE = 2299$, then find $m+n$, where $FP = \frac{m}{n}$ with m, n relatively prime positive integers.

Proposed by Michael Kural.

Answer. $\boxed{2440}$.

Solution. To deal with mixtillinear incircles, we utilize the following key lemma:

Lemma 27.1. *For a triangle ABC in general, suppose D lies on segment BC and a circle is tangent to segments AD, BC at X, Y and the circumcircle of ABC internally. Then the incenter I of ABC lies on line XY.*

Proof. Well known; see, for example, Lemma 6 at `http://yufeizhao.com/olympiad/geolemmas.pdf`. $\square$

Now, let line CD intersect Γ at S, T, and let U, V be the incenters of triangles AST, BST, respectively. By Lemma 1, U lies on WX and V lies on YZ. Let Q, R be the excenters of triangle EST across from T, S respectively. By an external analogue of Lemma 1, Q lies on WX as well, and R lies on YZ. Thus P can be recharacterized as the intersection of QU and RV.

Let K be the midpoint of arc ST on Γ containing E. Let ω be a circle centered at K passing through S and T. By a well known result, Q, U, V, R also lie on ω. Of course, we also have that Q, R lie on KE, the external bisector of $\angle SET$, and similarly U lies on KA, V lies on KB.

Now note that

$$\angle QRP = \angle KRV = 90^\circ - \frac{\angle RKB}{2} = 90^\circ - \frac{\angle EAB}{2}$$

and similarly $\angle RQP = 90^\circ - \frac{\angle EBA}{2}$. Thus $PRQ \sim FGH$, where FGH is the excentral triangle of ABE (F, G, H are the excenters across E, A, B respectively).

In particular, let M be the midpoint of arc AB on Γ containing E. M is the second intersection of the nine point center of FGH with GH, so M is the midpoint of GH. K is the center of ω passing through Q, R, so K is the midpoint of RQ. Thus K, M are corresponding points on similar triangles FGH and PRQ. In particular,

$$\angle FME = \angle FMH = \angle PKQ = \angle PKE = \angle PME$$

so M, P, F are collinear.

The rest of the problem is standard: we can easily compute $GH = 4576$, $FH = 3146$, and $FG = 3718$. Thus

$$FM = \sqrt{\frac{FG^2 + FH^2}{2} - \frac{GH^2}{4}} = 2574$$

and

$$FB = \frac{FH \cdot AB}{GH} = 374$$

so by Power of a Point

$$FP = \frac{FB \cdot FG/2}{FM} = \frac{2431}{9}$$

and the final answer is $2431 + 9 = 2440$. □

28. Find the number of ordered pairs $(P(x), Q(x))$ of polynomials with integer coefficients such that

$$P(x)^2 + Q(x)^2 = \left(x^{4096} - 1\right)^2.$$

Proposed by Michael Kural.

Answer. $\boxed{708588}$.

Solution. We claim that the answer for 2^n is $4 \cdot 3^{n-1}$ for $n \geq 1$. Note that the statement is equivalent to

$$(P(x) + iQ(x))(P(x) - iQ(x)) = (x-1)^2(x+1)^2 \prod_{k=0}^{n-2} (x^{2^k} + i)^2 (x^{2^k} - i)^2$$

thus we are decomposing the right-hand side into the product of a polynomial and its conjugate.

Lemma 28.1. *The polynomial $x^{2^k} - i$ is irreducible in $\mathbb{Z}[i]$, i.e. it is indecomposable over the Gaussian integers.*

Proof 1. Suppose

$$x^{2^k} - i = A(x)B(x)$$

for some $A, B \in \mathbb{Z}[i]$ of positive degree. Then

$$x^{2^k} + i = \bar{A(x)}\bar{B(x)}$$

so

$$x^{2^{k+1}} + 1 = (A(x)\bar{A(x)})(B(x)\bar{B(x)})$$

but $A(x)\bar{A(x)}$ and $B(x)\bar{B(x)}$ are integer polynomials with positive degree. However, $x^{2^{k+1}} + 1$ is irreducible, as it is a cyclotomic polynomial. Its irreducibility can also be shown by noting

$$(x+1)^{2^{k+1}} + 1 \equiv x^{2^{k+1}} + 2$$

where each coefficient is taken $\pmod 2$. So by Eisenstein's Criterion the shifted polynomial is irreduble, and so the original is irredubile, giving a contradiction. □

Proof 2. We now apply the same Eisenstein strategy to $x^{2^k} - i$ itself. Shift the polynomial by 1 to yield

$$(x+1)^{2^k} - i$$

Similarly to before, each coefficient in this polynomial is divisible by 2 except for x^{2^k} and the constant term, $1-i$. Now note that $1-i$ is irreducible in $Z[i]$ and $1-i \mid 2$, so by Eisenstein on $1-i$, this polynomial is irreducible. $\square$

Proof 3. By Gauss' Lemma it suffices to prove irreducibility in $\mathbb{Q}[i]$. If z is a root of $x^{2^k} - i$, then $\mathbb{Q}[z][i] = \mathbb{Q}[z]$ since i is a power of z. Also, z is a root of $x^{2^{k+1}} + 1$, which is irreducible in $\mathbb{Q}[x]$, as shown before.

$$2[\mathbb{Q}[z][i] : \mathbb{Q}[i]] = [\mathbb{Q}[z][i] : \mathbb{Q}[i]][\mathbb{Q}[i] : \mathbb{Q}] = [\mathbb{Q}[z][i] : \mathbb{Q}] = \mathbb{Q}[z][i] = 2^{k+1}$$

implying $[\mathbb{Q}[z][i] : \mathbb{Q}[i]] = 2^k$, so $x^{2^k} - i$ must be the minimal polynomial of z. $\square$

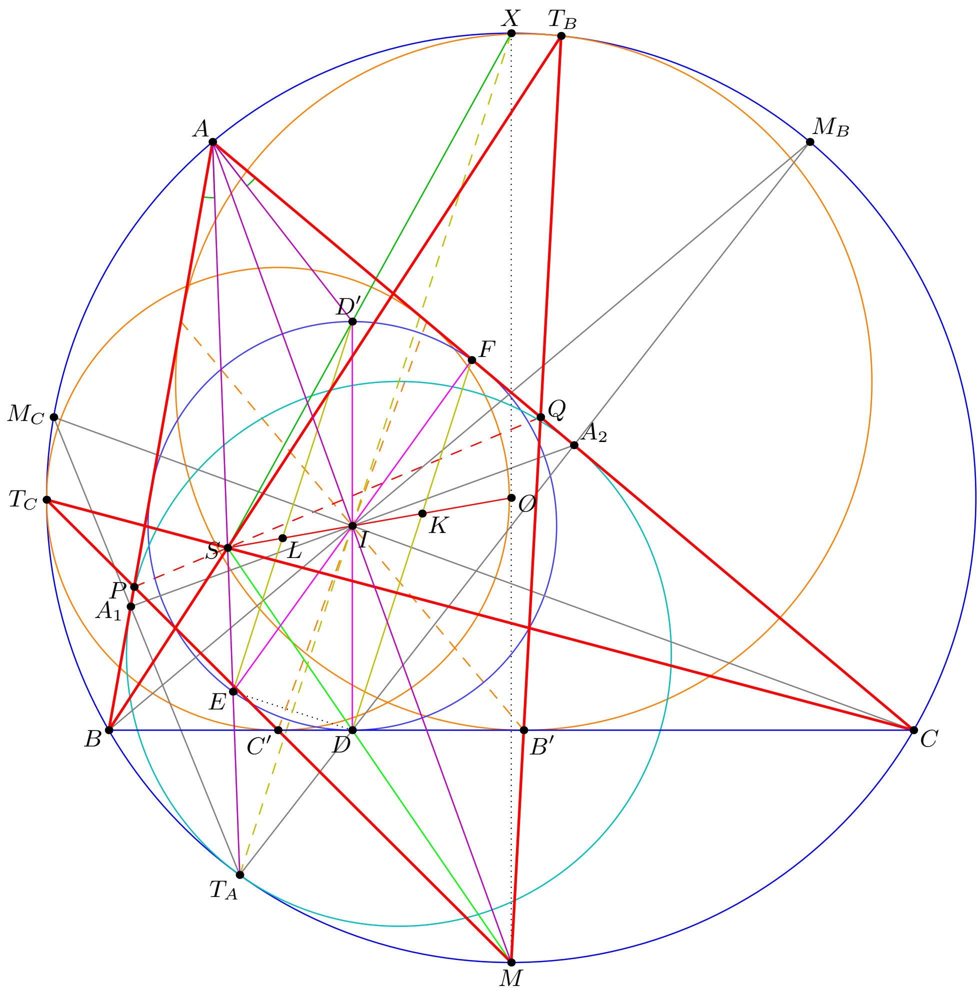
Now for each pair of conjugate polynomials in the RHS, one must become a factor of $P + Qi$ and one must become a factor of $P - Qi$. For each $(x^{2^k} + i)^2(x^{2^k} - i)^2$, there are three choices to do this: let $(x^{2^k} + i)^2$ divide $P + Qi$ and $(x^{2^k} - i)^2$ divide $P - Qi$, the reverse, or let $(x^{2^k} + i)(x^{2^k} - i)$ divide both. We necessarily must have $(x-1)(x+1)$ dividing both, so up to units there are 3^{n-1} choices for $P+Qi$. Including the units $1, -1, i, -i$ to multiply by, there are a total of $4 \cdot 3^{n-1}$ choices. $\square$

29. Let ABC be an acute scalene triangle with incenter I, and let M be the circumcenter of triangle BIC. Points D, B', and C' lie on side BC so that $\angle BIB' = \angle CIC' = \angle IDB = \angle IDC = 90^\circ$. Define $P = \overline{AB} \cap \overline{MC'}$, $Q = \overline{AC} \cap \overline{MB'}$, $S = \overline{MD} \cap \overline{PQ}$, and $K = \overline{SI} \cap \overline{DF}$, where segment EF is a diameter of the incircle selected so that S lies in the interior of segment AE. It is known that $KI = 15x$, $SI = 20x + 15$, $BC = 20x^{5/2}$, and $DI = 20x^{3/2}$, where $x = \frac{a}{b}(n + \sqrt{p})$ for some positive integers a, b, n, p, with p prime and $\gcd(a, b) = 1$. Compute $a + b + n + p$.

    Proposed by Evan Chen.

    Answer. $\boxed{99}$.

    Solution. The length of $BC = 20x^{5/2}$ is extraneous and not used below.



Define the A-*mixtilinear incircle* ω_A to be the circle internally tangent to the circumcircle of ABC at a point T_A while simultaneously tangent to $\overline{AB}$ and $\overline{AC}$. Define ω_B, ω_C, T_B and T_C simultaneously. Finally let ω and Ω denote the incircle and circumcircle of $\triangle ABC$.

Let M_B and M_C be the midpoints of the arcs $\widehat{AC}$ and $\widehat{AB}$ of Ω which do not contain the opposite vertices, and let the A-mixtilinear incircle be tangent to sides AB and AC at points A_1 and A_2. Also, observe that M is the midpoint of the arc $\widehat{BC}$ of Ω not containing A, a well-known fact which can be established by angle chasing.

First, we claim that points

$$\angle AIA_1 = \angle AIA_2 = 90^\circ.$$

Indeed, note that the homothety taking ω_A to Ω sends A_1 to M_C and A_2 to M_B. Therefore, applying Pascal's Theorem on hexagon

$$ABT_BMT_CC \quad (\text{on } \Omega)$$

implies that A_1, A_2, I are collinear. The conclusion is immediate.[2] By applying similar logic we discover that B' and C' are the contact points of ω_B and ω_C on side BC. Consequently, we discover that point M is the intersection of lines T_CC' and T_BB'.

Next, let X_{56} denote the center of the positive homothety taking the incircle to Ω.[3] We claim that A,

[2] The special case of this with $AB = AC$ appeared on IMO 1978.

[3] That is the actual name of this point – it is the 56th triangle center in Kimberling's Encyclopedia. Moreover, it is the isogonal conjugate of the Nagel point, as we will soon see.

X_{56}, and T_A are collinear.[4] Indeed, consider the composition of homotheties

$$\omega \xrightarrow{A} \omega_A \xrightarrow{T_A} \Omega.$$

By definition, it has center X_{56}. But the line AT_A is fixed by it, proving the claim. Also, note that D and M are images under the composed homothety. Hence, lines AT_A, BT_B, CT_C, DM and IO concur at X_{56}.

Next, by Pascal's Theorem on hexagon

$$ABT_BMT_CC \quad (\text{on } \Omega)$$

we find that P, X_{56}, Q are collinear. It follows that

$$X_{56} = S.$$

Now, let D' be the point diametrically opposite D on ω and X the point diametrically opposite M on Ω. Of course, points S, D', X are collinear as the homothety maps X to D'. We claim now that T_A, I, X are collinear[5]. Indeed, note that rays TA and TI are the symmedian and median of $\triangle T_AA_1A_2$, respectively; the result now follows since lines AX and M_BM_C are parallel.

Denote by L the intersection of lines $D'E$ and SI. From the collinearity above, we find that under the homothety from ω to Ω at S maps the point L to I. Moreover, we find that $D'EDF$ is a rectangle inscribed in ω by construction; thus $LI = IK$.

In summary, there is a homothety centered at S which sends

- ω to Ω,
- I to O,
- and L to I.

Let h be the homothety of this ratio. First, we have

$$h = \frac{SI}{SL} = \frac{SO}{SI} = \frac{IO}{LI}$$

but also

$$IO^2 = R(R-2r) = h(h-2)r^2.$$

In this way we obtain that

$$IO^2 = LI^2 \cdot h^2 = h(h-2)r^2$$

so

$$LI^2 = \left(1 - \frac{2}{h}\right) \cdot r^2.$$

Substituting the known values $LI = KI = 15x$ and $SI = 20x + 15$, and $r^2 = DI^2 = 400x^3$, we get

$$(15x)^2 = \left(1 - 2 \cdot \frac{5x+15}{20x+15}\right) \cdot 20^2 \cdot x^3.$$

Simplifying, we get

$$\frac{9}{16x} = 1 - 2 \cdot \frac{x+3}{4x+3} = \frac{2x-3}{4x+3}$$

id est,

$$32x^2 - 84x - 27 = 0.$$

Applying the quadratic formula gives and taking the positive root gives that

$$x = \frac{1}{64}\left(84 + 12\sqrt{73}\right) = \frac{3}{16}\left(7 + \sqrt{73}\right)$$

and the answer is $3 + 16 + 7 + 73 = 99$. □

[4] This is a particular instance of a theorem called Monge's Theorem. The proof is basically the same as the one we gave here.
[5] This appeared on an Iran 2002 olympiad.

Remark. In light of $x \approx 2.91$, we can compute the inradius and circumcircle are approximately 99.5 and 246.6. Moreover, $BC \approx 290.0$, though as stated before the side length of BC is irrelevant to the problem.

30. Let S be the value of

$$\sum_{n=1}^{\infty} \frac{d(n) + \sum_{m=1}^{\nu_2(n)} (m-3) d\left(\frac{n}{2^m}\right)}{n},$$

where $d(n)$ is the number of divisors of n and $\nu_2(n)$ is the exponent of 2 in the prime factorization of n. If S can be expressed as $(\ln m)^n$ for positive integers m and n, find $1000n + m$.

Proposed by Robin Park.

Answer. $\boxed{2004}$.

Solution. Note that the series is conditionally convergent ($\sum_{n=1}^{\infty} \frac{d(n)}{n}$ diverges), so we introduce a regulator s in our sum:

$$f(s) = \sum_{n=1}^{\infty} \frac{d(n) + \sum_{m=1}^{\nu_2(n)} (m-3) d\left(\frac{n}{2^m}\right)}{n^s}.$$

We now use the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. Then

$$\begin{aligned} f(s) &= \sum_{n=1}^{\infty} \frac{d(n)}{n^s} + \sum_{n=1}^{\infty} \frac{\sum_{m=1}^{\nu_2(n)} (m-3) d\left(\frac{n}{2^m}\right)}{n^s} \\ &= \zeta(s)^2 \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \cdots \right)^2 \\ &= \zeta(s)^2 \left(1 - \frac{1}{-1 + 2^s} \right)^2 \\ &= \zeta(s)^2 \left(\frac{-2 + 2^s}{-1 + 2^s} \right)^2. \end{aligned}$$

Now we take the limit of $f(s)$ as $s \to 1$. The denominator $-1 + 2^s$ vanishes, so we have to find $\lim_{s \to 1} \zeta(s)^2 (-2 + 2^s)^2$. It's pretty difficult to use l'Hopital's rule on the zeta function itself, so we use the Laurent expansion of $\zeta(s)$. It is well-known that the zeta function has a simple pole at $s = 1$, and by checking the coefficient we know that

$$\zeta(s) = \frac{1}{s-1} + c_0 + c_1(s-1) + c_2(s-1)^2 + \cdots.$$

So

$$\begin{aligned} \lim_{s \to 1} \zeta(s)^2 (-2 + 2^s)^2 &= \lim_{s \to 1} \left(\frac{1}{s-1} + c_0 + c_1(s-1) + c_2(s-1)^2 + \cdots \right)^2 (-2 + 2^s)^2 \\ &= \lim_{s \to 1} \frac{(-2 + 2^s)^2}{(s-1)^2} = \left(\lim_{s \to 1} \frac{-2 + 2^s}{s-1} \right)^2 = \left(\lim_{s \to 1} \frac{2^s \ln 2}{1} \right)^2 = (\ln 4)^2. \quad \square \end{aligned}$$