The Online Math Open Spring Contest
Official Solutions
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1. In English class, you have discovered a mysterious phenomenon – if you spend n hours on an essay, your score on the essay will be $100\left(1-4^{-n}\right)$ points if $2n$ is an integer, and 0 otherwise. For example, if you spend 30 minutes on an essay you will get a score of 50, but if you spend 35 minutes on the essay you somehow do not earn any points.

   It is 4AM, your English class starts at 8:05AM the same day, and you have four essays due at the start of class. If you can only work on one essay at a time, what is the maximum possible average of your essay scores?

   Proposed by Evan Chen.

   Answer. $\boxed{75}$.

   Solution. Note that the essay scores represent diminishing marginal returns – the number of additional points you gain with each additional half hour decreases. Therefore, your sum of scores is maximized if you distribute your time equally among all the essays.

   In the problem, there are eight half-hours to distribute among four essays, so one should spend an hour on each. This nets 75 points per essay and hence a final average of 75. □

2. Consider two circles of radius one, and let O and O' denote their centers. Point M is selected on either circle. If $OO' = 2014$, what is the largest possible area of triangle OMO'?

   Proposed by Evan Chen.

   Answer. $\boxed{1007}$.

   Solution. Observe that the base $\overline{OO'}$ is fixed at 2014, so we wish to maximize the height. Because M lies within one of either one of the centers, that means the height is at most 1 (and this is clearly achievable). So, the maximal area is $\frac{1}{2}\cdot 2014 = 1007$. □

3. Suppose that m and n are relatively prime positive integers with $A = \frac{m}{n}$, where

   $$A = \frac{2+4+6+\cdots+2014}{1+3+5+\cdots+2013} - \frac{1+3+5+\cdots+2013}{2+4+6+\cdots+2014}.$$

   Find m. In other words, find the numerator of A when A is written as a fraction in simplest form.

   Proposed by Evan Chen.

   Answer. $\boxed{2015}$.

   Solution. Let $N = 1007$. Observe that

   $$2+4+6+\cdots+2N = 2\left(1+2+\cdots+N\right) = N(N+1)$$

   while

   $$1+3+\cdots+(2N-1) = N^2.$$

   Indeed, these can all be proven by simply using the standard formula for an arithmetic series. Hence, the fraction in question is simply

   $$\frac{N+1}{N} - \frac{N}{N+1} = \frac{2N+1}{N(N+1)} = \frac{2015}{1007\cdot 1008}.$$

   Because $\gcd(2015, 1007) = \gcd(2015, 1008) = 1$, the answer is just 2015. □

4. The integers $1, 2, \dots, n$ are written in order on a long slip of paper. The slip is then cut into five pieces, so that each piece consists of some (nonempty) consecutive set of integers. The averages of the numbers on the five slips are 1234, 345, 128, 19, and 9.5 in some order. Compute n.

   Proposed by Evan Chen.

Answer. $\boxed{2014}$.

Solution. Note that the median is simply half the sum of the smallest and largest numbers on the piece. Furthermore, we can sort the piece in the order of the medians.

This means the first piece (containing 1) has largest number $2 \cdot 9.5 - 1 = 18$; in other words, the first of the pieces contains $1, 2, \dots, 18$. Then the next piece contains just the number 19. Similarly, the third piece contains the numbers 20 through 236 (as $\frac{1}{2}(20 + 236) = 128$); the fourth slip contains the numbers 237 through 453, and the last slip contains numbers 454 to 2014. Hence, $n = 2014$, the final answer. ☐

5. Joe the teacher is bad at rounding. Because of this, he has come up with his own way to round grades, where a *grade* is a nonnegative decimal number with finitely many digits after the decimal point.

   Given a grade with digits $a_1a_2 \dots a_m.b_1b_2 \dots b_n$, Joe first rounds the number to the nearest 10^{-n+1}th place. He then repeats the procedure on the new number, rounding to the nearest 10^{-n+2}th, then rounding the result to the nearest 10^{-n+3}th, and so on, until he obtains an integer. For example, he rounds the number 2014.456 via $2014.456 \to 2014.46 \to 2014.5 \to 2015$.

   There exists a rational number M such that a grade x gets rounded to at least 90 if and only if $x \geq M$. If $M = \frac{p}{q}$ for relatively prime integers p and q, compute $p + q$.

   Proposed by Yang Liu.

   Answer. $\boxed{814}$.

   Solution. The main idea is that the smallest grades which round to a 90 are precisely those of the form
   $$89.\underbrace{4444\dots44}_{n \text{ digits}}5.$$
   As n grows arbitrarily large, this limit approaches $89 + \frac{4}{9} = \frac{805}{9}$. So the answer is $805 + 9 = 814$. Note that M itself is not a grade (since grades have finitely many digits after the decimal point), so the distinction between $x \geq M$ and $x > M$ is not relevant. ☐

6. Let L_n be the least common multiple of the integers $1, 2, \dots, n$. For example, $L_{10} = 2{,}520$ and $L_{30} = 2{,}329{,}089{,}562{,}800$. Find the remainder when L_{31} is divided by 100,000.

   Proposed by Evan Chen.

   Answer. $\boxed{46800}$.

   Solution. Because 31 is prime, we have $L_{31} = 31L_{30}$. Hence, it suffices to compute the remainder when $31 \cdot 62800$ is divided by 10^5, which is 46800. ☐

7. How many integers n with $10 \leq n \leq 500$ have the property that the hundreds digit of $17n$ and $17n+17$ are different?

   Proposed by Evan Chen.

   Answer. $\boxed{84}$.

   Solution. Let A_n denote the value when the last two digits of $17n$ are deleted. Notice that $A_{n+1} - A_n$ is either 0 or 1 for every n. Hence, the problem is just asking for the number of n with $10 \leq n \leq 500$ such that $A_{n+1} - A_n = 1$.

   As n ranges from 10 to 500, the smallest number is $17 \cdot 10 = 170$ and the largest number is $17(500) + 17 = 8517$. In other words, $A_{10} = 1$ and $A_{501} = 85$. Hence for $10 \leq n \leq 500$, the value of A_n is increased exactly 84 times. Hence, the answer is 84. ☐

8. Let a_1, a_2, a_3, a_4, a_5 be real numbers satisfying

$$\begin{aligned} 2a_1 + a_2 + a_3 + a_4 + a_5 &= 1 + \tfrac{1}{8}a_4 \\ 2a_2 + a_3 + a_4 + a_5 &= 2 + \tfrac{1}{4}a_3 \\ 2a_3 + a_4 + a_5 &= 4 + \tfrac{1}{2}a_2 \\ 2a_4 + a_5 &= 6 + a_1 \end{aligned}$$

Compute $a_1 + a_2 + a_3 + a_4 + a_5$.

Proposed by Evan Chen.

Answer. $\boxed{2}$.

Solution. Add eight times the first equation, four times the second, and two times the third to the fourth. We obtain that

$$16a_1 + 16a_2 + 16a_3 + 16a_4 + 15a_5 = 30 + a_4 + a_3 + a_2 + a_1.$$

Letting S be the desired sum, we readily derive $16S = 30 + S$, so $S = 2$. □

9. Eighteen students participate in a team selection test with three problems, each worth up to seven points. All scores are nonnegative integers. After the competition, the results are posted by Evan in a table with 3 columns: the student's name, score, and rank (allowing ties), respectively. Here, a student's rank is one greater than the number of students with strictly higher scores (for example, if seven students score $0, 0, 7, 8, 8, 14, 21$ then their ranks would be $6, 6, 5, 3, 3, 2, 1$ respectively).

When Richard comes by to read the results, he accidentally reads the rank column as the score column and vice versa. Coincidentally, the results still made sense! If the scores of the students were $x_1 \le x_2 \le \cdots \le x_{18}$, determine the number of possible values of the 18-tuple $(x_1, x_2, \ldots, x_{18})$. In other words, determine the number of possible multisets (sets with repetition) of scores.

Proposed by Yang Liu.

Answer. $\boxed{131072}$.

Solution. Let $n = 18$ and suppose $1 = r_1 \le r_2 \le \cdots \le r_n$ is a set of ranks for the students. The main observation is that every such set of ranks gives rise to exactly one set of scores. Explicitly, we simply set the score of the nth student as 1, and then work backwards to construct the set of scores for the students. Because there are $21 \ge n$ points available on the contest, this construction will always terminate successfully.

Hence, the problem is equivalent to counting the number of tuples of ranks. Evidently $r_1 = 1$. For each subsequent i, either $r_{i+1} = r_i$, or r_{i+1} is r_i plus the number of indices j with $r_i = r_j$. In other words, we either stay the same or increase. This means that we have 2 choices at each step, and the answer is $2^{n-1} = 2^{17} = 131072$. □

10. Let $A_1A_2 \ldots A_{4000}$ be a regular 4000-gon. Let X be the foot of the altitude from A_{1986} onto diagonal $A_{1000}A_{3000}$, and let Y be the foot of the altitude from A_{2014} onto $A_{2000}A_{4000}$. If $XY = 1$, what is the area of square $A_{500}A_{1500}A_{2500}A_{3500}$?

Proposed by Evan Chen.

Answer. $\boxed{2}$.

Solution. Let ω denote the circumcircle of the 4000-gon and let O denote its center. One can verify that $OXA_{2014}Y$ is a rectangle, hence $1 = XY = OA_{2014}$ is a radius ω. Consequently, the side length of the square is $\sqrt{2}$ and the area is $\left(\sqrt{2}\right)^2 = 2$. □

11. Let X be a point inside convex quadrilateral $ABCD$ with $\angle AXB + \angle CXD = 180^\circ$. If $AX = 14$, $BX = 11$, $CX = 5$, $DX = 10$, and $AB = CD$, find the sum of the areas of $\triangle AXB$ and $\triangle CXD$.

   Proposed by Michael Kural.

   Answer. $\boxed{90}$.

   Solution. Because $AB = CD$, we can construct a point X' outside of $ABCD$ such that $\triangle AX'B \cong CXD$. Additionally,
$$\angle AXB + \angle AX'B = \angle AXB + \angle CXD = 180$$
so $AX'BX$ is cyclic, and it's sufficient to find the area of this quadrilateral. Let O be the center of its circumcircle. Note that we can rearrange the triangles $OAX', OX'B, OBX, OXA$ while keeping each of the points equidistant from O, (so they are still cyclic) and not changing the total area of $AX'BX$. The side lengths of $AX'BX$ are $10, 5, 11, 14$, so we rearrange the lengths to be in the order $14, 5, 11, 10$. But $14^2 + 5^2 = 11^2 + 10^2 = 221$, so this new cyclic quadrilateral is composed of two right triangles with common hypotenuse $\sqrt{221}$. Thus its total area is
$$\frac{1}{2} \cdot 14 \cdot 5 + \frac{1}{2} \cdot 11 \cdot 10 = 90. \qquad \square$$

12. The points A, B, C, D, E lie on a line ℓ in this order. Suppose T is a point not on ℓ such that $\angle BTC = \angle DTE$, and $\overline{AT}$ is tangent to the circumcircle of triangle BTE. If $AB = 2$, $BC = 36$, and $CD = 15$, compute DE.

   Proposed by Yang Liu.

   Answer. $\boxed{954}$.

   Solution. By simple angle chasing, we find that
$$\angle ATC = \angle ATB + \angle BTC = \angle TEB + \angle DTE = \angle TCB.$$
Thus line AT is also a tangent to the circumcircle of triangle TEF. Hence
$$2AE = AB \cdot AE = AT^2 = AC \cdot AD = 38 \cdot 53 = 2014.$$
This implies $AE = 1007$, whence $DE = 1007 - 53 = 954$. $\square$

13. Suppose that g and h are polynomials of degree 10 with integer coefficients such that $g(2) < h(2)$ and
$$g(x)h(x) = \sum_{k=0}^{10} \left(\binom{k+11}{k} x^{20-k} - \binom{21-k}{11} x^{k-1} + \binom{21}{11} x^{k-1} \right)$$
holds for all nonzero real numbers x. Find $g(2)$.

   Proposed by Yang Liu.

   Answer. $\boxed{2047}$.

   Solution. Let $n = 10$. By the Hockey Stick identity, one discovers the factorization
$$p(x) = \left(x^n + x^{n-1} + x^{n-2} + \ldots + x + 1\right) \left(\sum_{i=0}^{n} \binom{n+i}{n} x^{n-i} \right).$$
The polynomial $x^{10} + x^9 + \cdots + 1$ happens to be irreducible (it is the 11th cyclotomic polynomial), and so this must be the unique factorization into two polynomials of degree n. Hence, the answer is just $1 + 2 + \cdots + 2^{10} = 2047$. $\square$

14. Let ABC be a triangle with incenter I and $AB = 1400$, $AC = 1800$, $BC = 2014$. The circle centered at I passing through A intersects line BC at two points X and Y. Compute the length XY.

    Proposed by Evan Chen.

    Answer. $\boxed{1186}$.

    Solution. Construct the points B' and C' on $\overline{BC}$ so that $CA = CB'$ and $BA = BC'$. These are consequently the reflections of B and C across lines CI and BI, respectively. Thus $AI = B'I = C'I$, and hence B and C are precisely the points X and Y. Then, it is straightforward to compute $XY = 1400 + 1800 - 2014 = 1186$. □

15. In Prime Land, there are seven major cities, labelled $C_0, C_1, \ldots, C_6$. For convenience, we let $C_{n+7} = C_n$ for each $n = 0, 1, \ldots, 6$; i.e. we take the indices modulo 7. Al initially starts at city C_0.

    Each minute for ten minutes, Al flips a fair coin. If the coin land heads, and he is at city C_k, he moves to city C_{2k}; otherwise he moves to city C_{2k+1}. If the probability that Al is back at city C_0 after 10 moves is $\frac{m}{1024}$, find m.

    Proposed by Ray Li.

    Answer. $\boxed{1171}$.

    Solution. Let is ignore for now the indices modulo 7. Then, the process described just selects an 10-digit binary number between 0 and $2^{10} - 1 = 1023$ inclusive. Furthermore, we end at C_0 if and only if the number is divisible by 7. There are 147 such multiples of 7, so the probability is $\frac{147}{1024}$, giving $147 + 1024 = 1171$. □

16. Say a positive integer n is *radioactive* if one of its prime factors is strictly greater than $\sqrt{n}$. For example, $2012 = 2^2 \cdot 503$, $2013 = 3 \cdot 11 \cdot 61$ and $2014 = 2 \cdot 19 \cdot 53$ are all radioactive, but $2015 = 5 \cdot 13 \cdot 31$ is not. How many radioactive numbers have all prime factors less than 30?

    Proposed by Evan Chen.

    Answer. $\boxed{119}$.

    Solution. Notice that counting N is just equivalent to counting multisets of primes such that one prime exceeds the product of the others. Let p be the largest prime, and put $N = px$. Note that $1 \le x \le p - 1$ and that any value of x is achievable in exactly one way (by unique prime factorization). Hence for a fixed value of p there are exactly $p - 1$ values of x possible, and hence $p - 1$ ways.

    The answer is thus
    $$\sum_{\substack{p \le 30 \\ p \text{ prime}}} p - 1 = 129 - 1 \cdot 10 = 119. \qquad \square$$

17. Let $AXYBZ$ be a convex pentagon inscribed in a circle with diameter $\overline{AB}$. The tangent to the circle at Y intersects lines BX and BZ at L and K, respectively. Suppose that $\overline{AY}$ bisects $\angle LAZ$ and $AY = YZ$. If the minimum possible value of
    $$\frac{AK}{AX} + \left(\frac{AL}{AB}\right)^2$$
    can be written as $\frac{m}{n} + \sqrt{k}$, where m, n and k are positive integers with $\gcd(m, n) = 1$, compute $m + 10n + 100k$.

    Proposed by Evan Chen.

    Answer. $\boxed{343}$.

Solution. Let O be the midpoint of $\overline{AB}$, and let $AB = 2r$. The key observation which will let us set up the algebra is that $\triangle LOY \sim \triangle YBK$. First, because $\angle YZA = \angle YAZ = \angle LAZ$, we find $\overline{LA}$ is a tangent to the semicircle. This follows because

$$\angle AOL = \frac{1}{2}\angle AOY = \angle OBY \implies \overline{LO} \parallel \overline{YB} \quad \text{and} \quad \angle AOY = \frac{1}{2}\angle AOZ = \angle OBZ \implies \overline{YO} \parallel \overline{ZB}.$$

So, we may set $YK = kh$ and $KB = kr$, where $AL = LY = h$.

First, we claim have $\frac{AK}{AX} = \frac{3}{2}k + 1$. By Ptolemy's Theorem on $ABKL$,

$$BL \cdot AK = AL \cdot BK + AB \cdot LK = 2(k+1)rh + khr = (3k+2)hr.$$

But $\frac{1}{2}[LAB] = BL \cdot AX = AL \cdot AB = h(2r)$ and the conclusion follows.

Next we claim $k = \frac{2r^2}{r^2+h^2}$. This follows from $LB^2 = AL^2 + AB^2 = KL^2 + KB^2$, whence $h^2 + 4r^2 = (k+1)^2h^2 + k^2r^2$. Then, $k(k+2)h^2 = (4-k^2)r^2$. Cancelling a factor of $2+k$ gives $kh^2 = (2-k)r^2$, so $k = \frac{2r^2}{r^2+h^2}$.

Thus, if we let $x = \frac{h}{r}$, we get that

$$\begin{aligned} \frac{AK}{AX} + \left(\frac{AL}{AB}\right)^2 &= \frac{3}{2}k + 1 + \left(\frac{h}{2r}\right)^2 \\ &= \frac{3}{1+x^2} + 1 + \frac{x^2}{4} \\ &= \frac{3}{4} + \frac{3}{1+x^2} + \frac{1+x^2}{4} \\ &\geq \frac{3}{4} + 2\sqrt{\frac{3}{4}} \\ &= \frac{3}{4} + \sqrt{3} \end{aligned}$$

and the answer is 343. $\square$

18. Find the number of pairs (m, n) of integers with $-2014 \leq m, n \leq 2014$ such that $x^3 + y^3 = m + 3nxy$ has infinitely many integer solutions (x, y).

Proposed by Victor Wang.

Answer. $\boxed{25}$.

Solution. This is equivalent to $(x+y+n)(x^2+y^2+n^2-xy-n(x+y)) = m + n^3$, so infinitely many solutions exist if and only if $m = -n^3$. This permits $-12 \leq m \leq 12$, giving 25 solutions. $\square$

19. Find the sum of all positive integers n such that $\tau(n)^2 = 2n$, where $\tau(n)$ is the number of positive integers dividing n.

Proposed by Michael Kural.

Answer. $\boxed{100}$.

Solution. Clearly, we may write $n = 2u^2$, where u is some positive integer. Now note that the function

$$g(n) = \frac{\tau(n)^2}{n}$$

is multiplicative.

Observe that the exponent of 2 in n is odd, so we find that $g(2) = g(8) = 2$, but $g(2^j) < 2$ for $j \geq 5$. Moreover, the exponent of any prime p (other than 2) in n is even, and we can check $g\left(p^{2k}\right) \leq 1$ with equality precisely when $p = 3$ and $k \in \{0, 1\}$. So $g(n) \leq 2$, for all n, with equality at $n = 2, 8, 18, 72$. Hence the answer is $2 + 8 + 18 + 72 = 100$. $\square$

20. Let ABC be an acute triangle with circumcenter O, and select E on $\overline{AC}$ and F on $\overline{AB}$ so that $\overline{BE} \perp \overline{AC}$, $\overline{CF} \perp \overline{AB}$. Suppose $\angle EOF - \angle A = 90^\circ$ and $\angle AOB - \angle B = 30^\circ$. If the maximum possible measure of $\angle C$ is $\frac{m}{n} \cdot 180^\circ$ for some positive integers m and n with $m < n$ and $\gcd(m,n) = 1$, compute $m + n$.

Proposed by Evan Chen.

Answer. $\boxed{47}$.

Solution. The key is the following lemma: if O lies inside triangle AEF and $\angle EOF - \angle A = 90^\circ$ then $\angle A = 45^\circ$. Indeed, the conditions imply that pentagon $BEOFC$ is cyclic, so $\angle BOC = 90^\circ$.

Simple calculations give that $\angle C = 70^\circ - \frac{1}{3}\angle A$. So we wish to minimize $\angle A$. Note that if $\angle A \le 45^\circ$, then $\angle BOC \le 90^\circ$ which implies that O lies outside the circle with diameter $\overline{BC}$ while inside ABC, which causes the lemma to apply. In other words $\angle A \le 45^\circ \implies \angle A = 45^\circ$. So the minimum possible value of $\angle C$ is 55°. This is achieved if $(\angle A, \angle B, \angle C) = (45^\circ, 80^\circ, 55^\circ)$.

Converting to radians, $55^\circ = \frac{11}{36}\pi$ and $11 + 36 = 47$. □

21. Let $b = \frac{1}{2}(-1 + 3\sqrt{5})$. Determine the number of rational numbers which can be written in the form

$$a_{2014}b^{2014} + a_{2013}b^{2013} + \cdots + a_1 b + a_0$$

where $a_0, a_1, \dots, a_{2014}$ are nonnegative integers less than b.

Proposed by Michael Kural and Evan Chen.

Answer. $\boxed{15}$.

Solution. Let $c_n = \frac{1}{3\sqrt{5}}\left(b^n - \overline{b}^n\right)$, where $\overline{b} = \frac{1}{2}\left(-1 - 3\sqrt{5}\right)$.

It is easy to derive that $c_0 = 0$, $c_1 = 1$, and $c_n = -c_{n-1} + 11c_{n-2}$ for every positive integer n. The first few terms of the sequence are $0, 1, -1, 12, -23, 155, -408, 2113, \dots$.

Then, the problem is equivalent to selecting the integers a_i so that $a_0c_0 + a_1c_1 + \cdots + a_nc_n = 0$. This is because if $\sum_n a_n b^n$ is rational, so is its conjugate, implying the difference $\sum a_n b^n - a_n \overline{b}^n$ is zero.

It's not hard to verify that $|c_n| > 2\,|c_{n-1} + c_{n-3} + \dots|$ for each $n \ge 5$. Hence we only need to consider the case where $a_5 = a_6 = \cdots = 0$. This is simple casework (noting that 12 and -23 are overwhelmingly the largest terms), and the solutions are

$$(a_1, a_2, a_3, a_4) \in \{(0,0,0,0), (1,1,0,0), (2,2,0,0), (0,1,2,1), (1,2,2,1)\}.$$

where a_0 can be arbitrary (as $c_0 = 0$). Hence there are 15 such 5-tuples. □

22. Let $f(x)$ be a polynomial with integer coefficients such that $f(15)f(21)f(35) - 10$ is divisible by 105. Given $f(-34) = 2014$ and $f(0) \ge 0$, find the smallest possible value of $f(0)$.

Proposed by Michael Kural and Evan Chen.

Answer. $\boxed{620}$.

Solution. Let $p = 3$, $q = 5$, and $r = 7$. Note that

$$f(pq)f(qr)f(rp) = f(pq + qr + rp)f(0)^2 \pmod{pqr}.$$

This follows from the Chinese Remainder Theorem; we consider the equation modulo each of p, q, r. For this particular problem, we see $2014 \equiv f(-34) \equiv f(15 + 21 + 34) \pmod{105}$ and $f(15)f(21)f(31) \equiv 10 \pmod{105}$.

Now, let $x = f(0)$. Accordingly we find $x^2 \equiv 10 \cdot 2014^{-1} \equiv (-95) \cdot 19^{-1} \equiv -5 \pmod{105}$. The solutions are $x \equiv 10, 25, 80, 95 \pmod{105}$ by the Chinese Remainder Theorem.

In addition, we require $x \equiv f(-34) \equiv 2014 \pmod{34}$; that is, we also need $x \equiv 8 \pmod{34}$. In short we must find the smallest solution for

$$x \equiv 10, 25, 80, 95 \pmod{105} \quad \text{and} \quad x \equiv 8 \pmod{34}$$

The $x \equiv 8 \pmod{34}$ becomes $x \equiv 110 \pmod{170}$ (since $5 \mid x$). We then discover that $x = 620$ is the smallest working value (after trying $110, 280, 450$). A working construction is $f(x) = -41x + 620$.

Hence, the minimum possible value of $f(0)$ is 620. □

23. Let Γ_1 and Γ_2 be circles in the plane with centers O_1 and O_2 and radii 13 and 10, respectively. Assume $O_1O_2 = 2$. Fix a circle Ω with radius 2, internally tangent to Γ_1 at P and externally tangent to Γ_2 at Q . Let ω be a second variable circle internally tangent to Γ_1 at X and externally tangent to Γ_2 at Y. Line PQ meets Γ_2 again at R, line XY meets Γ_2 again at Z, and lines PZ and XR meet at M.

As ω varies, the locus of point M encloses a region of area $\frac{p}{q}\pi$, where p and q are relatively prime positive integers. Compute $p + q$.

Proposed by Michael Kural.

Answer. $\boxed{16909}$.

Solution. Let O_3 be the center of Ω. Note that then P, O_3, O_1 are collinear and P, Q, O_2 are collinear. But O_3PQ and O_2QF are isosceles triangles, so

$$\angle O_3PF = \angle O_3PQ = \angle O_3QP = \angle O_2QF = \angle O_2FQ = \angle O_2FP$$

Thus $\overline{O_2F} \parallel \overline{O_3P}$. Let $\overline{O_1O_2}$ meet $\overline{PF}$ at S. Then since $\overline{O_1P} \parallel \overline{O_2F}$, we see S is the center of negative homothety mapping Γ_1 to Γ_2. So S lies on $\overline{PQ}$, and similarly S also lies on $\overline{XY}$. Now this negative homothety maps P to the intersection past S of $\overline{PS}$ with Γ_2 and X to the intersection past S of $\overline{XS}$ with Γ_2. But these intersections are R and Z, respectively. So this homothety maps $\overline{PX}$ to $\overline{RZ}$, and so $\overline{PX} \parallel \overline{RZ}$. Because this maps $\overline{PX}$ on circle O_1 to $\overline{ZR}$ on circle O_2, we obtain

$$\frac{PX}{ZR} = \frac{O_1P}{O_2R} = \frac{13}{10}.$$

Note that if lines PR and XZ meet at S inside quadrilateral $PXRZ$, then lines PZ and RX meet at M outside this quadrilateral. So $\frac{MZ}{MP} = \frac{10}{13}$, and $\frac{PM}{PZ} = \frac{13}{3}$. Now P is fixed, but as ω varies, Z can take be all points on circle Γ_2. So the locus of M is circle Γ_2, dilated about P with scale factor $\frac{13}{3}$. Thus its area is

$$\pi\left(\frac{13}{3}\cdot 10\right)^2 = \left(\frac{16900}{9}\right)\pi$$

and our answer is 16909. □

24. Let $\mathcal{P}$ denote the set of planes in three-dimensional space with positive x, y, and z intercepts summing to one. A point (x, y, z) with $\min\{x, y, z\} > 0$ lies on exactly one plane in $\mathcal{P}$. What is the maximum possible integer value of $\left(\frac{1}{4}x^2 + 2y^2 + 16z^2\right)^{-1}$?

Proposed by Sammy Luo.

Answer. $\boxed{21}$.

Solution. The points on the planes in P are of the form $(x, y, z) = (a_1e_1, a_2e_2, a_3e_3)$, where the a_i (axis intercepts) and e_i (weights) both sum to 1. Hence, by Cauchy Schwarz we have

$$1 = (a_1 + a_2 + a_3)(e_1 + e_2 + e_3) \geq (\sqrt{a_1e_1} + \sqrt{a_2e_2} + \sqrt{a_3e_3})^2.$$

This implies that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \leq 1.$$

Equality can be attained for any choice of x, y, z satisfying the equality by (and only by) setting $a_1 = e_1 = \sqrt{x}$ and so forth.

Thus we find S is the set of points with $\sqrt{x}+\sqrt{y}+\sqrt{z} = 1$. Then by Hölder (or weighted power mean), we have the inequality

$$\left(\frac{1}{8}x^2 + y^2 + 8z^2\right)\left(2 + 1 + \frac{1}{2}\right)^3 \geq \left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right)^4 = 1.$$

Therefore,

$$\left(\frac{1}{4}x^2 + 2y^2 + 16z^2\right)^{-1} \leq \frac{1}{2}\left(2 + 1 + \frac{1}{2}\right)^3 = \frac{343}{16} = 21 + \frac{7}{16}.$$

Since the values can be changed continuously, the largest possible integer value is 21. □

25. If

$$\sum_{n=1}^{\infty} \frac{\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}}{\binom{n+100}{100}} = \frac{p}{q}$$

for relatively prime positive integers p, q, find $p + q$.

Proposed by Michael Kural.

Answer. $\boxed{9901}$.

Solution. Let $H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$. Note that

$$\binom{n+100}{n} = \left(\frac{n+100}{100}\right)\binom{n+99}{99}$$

and

$$\binom{n+100}{n} = \left(\frac{n+1}{100}\right)\binom{n+100}{99}$$

so

$$\frac{1}{\binom{n+99}{99}} - \frac{1}{\binom{n+100}{99}} = \frac{\frac{99}{100}}{\binom{n+100}{100}}.$$

So the sum is

$$\frac{100}{99}\left(H_1\left(\binom{100}{99}^{-1} - \binom{101}{99}^{-1}\right) + H_2\left(\binom{101}{99}^{-1} - \binom{102}{99}^{-1}\right) + \cdots\right).$$

which equals

$$100/99\left(\binom{100}{99}^{-1} + \frac{1}{2}\binom{101}{99}^{-1} + \frac{1}{3}\binom{102}{99}^{-1} \cdots\right).$$

Note that $n\binom{99+n}{100} = 100\binom{99+n}{100}$, so this becomes

$$\frac{1}{99}\left(\binom{100}{100}^{-1} + \binom{101}{100}^{-1} + \binom{102}{100}^{-1} + \cdots\right).$$

Now again using the first identity, this telescopes and becomes

$$\frac{100}{99^2}\left(\binom{99}{99}^{-1} - \binom{100}{99}^{-1} + \binom{100}{99}^{-1} - \binom{101}{99}^{-1} \cdots\right) = \frac{100}{99^2}\left(\binom{99}{99}^{-1}\right) = \frac{100}{9801}.$$

yielding a final answer of $100 + 9801 = 9901$. Replacing 100 with a general k, the answer is $\frac{k}{(k-1)^2}$. □

26. Qing initially writes the ordered pair $(1, 0)$ on a blackboard. Each minute, if the pair (a, b) is on the board, she erases it and replaces it with one of the pairs $(2a - b, a)$, $(2a + b + 2, a)$ or $(a + 2b + 2, b)$. Eventually, the board reads $(2014, k)$ for some nonnegative integer k. How many possible values of k are there?

   Proposed by Evan Chen.

   Answer. $\boxed{720}$.

   Solution. Consider instead the pairs with both entries increased by one. Then the operation corresponds to starting with $(2, 1)$ and repeatedly replacing (m, n) by one of $(2m - n, m)$, $(2m + n, m)$ and $(m + 2n, n)$.

   We can make a few simple observations about the resulting pairs:

   - The first entry always exceeds the second entry.
   - There is always exactly one even entry.
   - The entries are always relatively prime to each other.

   The crucial claim is that in fact this is a complete characterization for all achievable pairs. The proof is a simple strong induction on $m + n$ where (m, n) has the above properties. Indeed, one can actually show there is a *unique* construction of (m, n) from $(2, 1)$ using the above operations.

   Hence, the answer is the number of even integers relatively prime to 2015. Letting φ denote Euler's totient function, this is just $\frac{1}{2}\varphi(2015) = \frac{1}{2} \cdot 4 \cdot 12 \cdot 30 = 720$. □

27. A frog starts at 0 on a number line and plays a game. On each turn the frog chooses at random to jump 1 or 2 integers to the right or left. It stops moving if it lands on a nonpositive number or a number on which it has already landed. If the expected number of times it will jump is $\frac{p}{q}$ for relatively prime positive integers p and q, find $p + q$.

   Proposed by Michael Kural.

   Answer. $\boxed{301}$.

   Solution. Let $g(n)$ be the number of ways for the frog to jump n times and land on a positive number that it has not landed on before. Let $f(n, k)$ be the number of ways to do this such that its first k jumps are 2 to the right and its $k + 1$'th is not, for $0 \le k \le n$. (We define $f(n, n) = 1$ for the one path that consists of n jumps 2 to the right.) Note that

$$g(n) = f(n, 0) + f(n, 1) + \cdots + f(n, n)$$

   for $n \ge 0$.

   Note $f(n, 0) = g(n-1)$ for $n \ge 1$, since the frogs first jump must be 1 to the right, and starting from 1 with 0 already landed on is equivalent to starting from 0 again with one less step taken. Additionally, $f(n, n - 1) = 2$ for $n \ge 2$, since after moving $n - 1$ steps to the right, the frog can either move one to the left or one to the right.

   Let $1_{a \ge b}$ be defined as taking the value 1 if $a \ge b$ and 0 otherwise. Consider the value of $f(j + k, k)$ for $j \ge 2$, $k \ge 1$. After landing on the numbers $2, 4, \cdots 2k$, the possible remaining numbers to land on are $1, 3, \cdots 2k - 1$, and $2k + 1, 2k + 2, 2k + 3, \cdots$. It is not hard to see that the frog can take 4 possible paths starting at $2k$ and not moving to $2k + 2$ initially:

   - The frog can move to $2k + 1$ and remain to the right of $2k$ for the next $j - 1$ steps. This is equivalent to starting at 0 and making $j - 1$ arbitrary legal steps, so the number of ways to do this is $g(j - 1)$.
   - The frog can move to $2k - 1$, and then continue to $2k - 3, 2k - 5$, etc. until it stops at a positive odd integer. In order for this to be possible, we must have $j \le k$, so the number of ways to do this is $1_{j \ge k}$.

- The frog can move to $2k-1$, then hop to the right again to $2k+1$, and remain to the right of $2k$ for the next $j-2$ steps. This is possible given $k \geq 1, j \geq 2$, and similarly to the first case, the number of ways to do this is $g(j-2)$.
- The frog can move to $2k+1$, then hop left to $2k-1$ and continue moving along the path $2k-1, 2k-3, \cdots$. The number of ways to do this is $1_{j \leq k+1}$.

Thus if we substitute $m = j+k$, we obtain

$$f(m,k) = g(m-k-1) + g(m-k-2) + 1_{2k \geq m} + 1_{2k+1 \geq m}$$

for $m \geq k+2$ and $k \geq 1$. Substituting this into the definition of $g(m)$ yields

$$\begin{aligned} g(m) &= g(m-1) + (g(m-2)+g(m-3)) + \cdots + (g(1)+g(0)) + 2 + 1 \\ &\quad + \sum_{m/2 \leq k \leq m-2} 1 + \sum_{(m-1)/2 \leq k \leq m-2} 1 \\ &= g(m-1) + g(m-2) + 2g(m-3) + \cdots + 2g(1) + g(0) + 3 + \lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil - 2 \\ &= g(m-1) + g(m-2) + 2g(m-3) + \cdots + 2g(1) + g(0) + m + 1 \end{aligned}$$

for all $m \geq 2$. This implies

$$g(m) = g(m-1) + 2g(m-3) + 1$$

for all $m \geq 3$.

Let $h(n)$ be the probability that the frog makes $n-1$ legal jumps and then must stop after its nth jump. The expected value of the amount of jumps the frog makes is

$$E = \sum_{n \geq 0} n \cdot h(n).$$

But note that

$$h(0) + h(1) + \cdots + h(k) + \frac{g(k)}{4^k} = 1$$

and

$$\begin{aligned} 0 \cdot h(0) + \cdots + k \cdot h(k) &= (k)(h(0) + \cdots h(k)) - (h(0)) - (h(0)+h(1)) - \cdots - (h(0) + \cdots + h(k-1)) \\ &= k(1 - \frac{g(k)}{4^k}) - (1 - \frac{g(0)}{4^0}) - \cdots - (1 - \frac{g(k-1)}{4^{k-1}} \\ &= \left(\sum_{0 \leq n < k} \frac{g(k)}{4^k} \right) - k \cdot \frac{g(k)}{4^k}. \end{aligned}$$

But the last term approaches 0 since $g(k) \leq 3^k$, so taking the limit to infinity yields

$$E = \sum_{n \geq 0} \frac{g(k)}{4^k}.$$

Thus

$$E - \frac{2E}{4} - \frac{E}{4^3} = \frac{1}{4^0} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots$$

and

$$\frac{31E}{64} = \frac{4}{3} - \frac{1}{4} = \frac{13}{12} \Rightarrow E = \frac{208}{93}.$$

This gives the answer of $208 + 93 = 301$. $\square$

28. In the game of Nim, players are given several piles of stones. On each turn, a player picks a nonempty pile and removes any positive integer number of stones from that pile. The player who removes the last stone wins, while the first player who cannot move loses.

    Alice, Bob, and Chebyshev play a 3-player version of Nim where each player wants to win but avoids losing at all costs (there is always a player who neither wins nor loses). Initially, the piles have sizes 43, 99, x, y, where x and y are positive integers. Assuming that the first player loses when all players play optimally, compute the maximum possible value of xy.

    Proposed by Sammy Luo.

    Answer. $\boxed{7800}$.

    Solution. Call a game position an A-position if it results in a win for the next player, a C-position if it results in a loss, and a B-position otherwise. We use the following definitions to make thing simpler: For a nonnegative integer $a = \overline{a_n a_{n-1} \cdots a_1 a_0}_2$ written in base 2, define the ternation of a, a_T, to be $\overline{a_n a_{n-1} \cdots a_1 a_0}_3$, i.e. the number that results when the base 2 representation of a is interpreted as a base 3 integer. Define the trim-sum of two numbers $a = \overline{a_n a_{n-1} \cdots a_1 a_0}_3$ and $b = \overline{b_n b_{n-1} \cdots b_1 b_0}_3$ written in base 3 (possibly with leading zeroes), represented by $a \boxplus b$, such that

    $$a \boxplus b = \overline{c_n c_{n-1} \cdots c_1 c_0}_3$$

    where $c_i \equiv a_i + b_i \pmod 3$ is 0, 1 or 2.

    The key is the following proposition. A position $P = (x_1, x_2, \cdots, x_n)$ in 3-Nim is a C-position if and only if

    $$(x_1)_T \boxplus (x_2)_T \boxplus \cdots \boxplus (x_n)_T = 0.$$

    In this case, we want $101011_3 \boxplus 1100011_3 \boxplus x_T \boxplus y_T = 0$, so we need $x_T \boxplus y_T = 2102011_3$. Since the base 3 digits in x_T and y_T are all zero or one, we see that $x_T + y_T = 2102011_3$, so to maximize their product, we make them as close as possible. This is done by assigning the first occurrence of a 1 in the sum's base 3 representation to x and every other occurrence of a 1 to y, yielding $x = 1101000_2 = 104, y = 1001011_2 = 75$. These multiply to give 7800.

    The proof of the key result used can actually generalize to more than 3 players, but for simplicity we'll just include the 3-player version. We use strong induction on $s = x_1 + x_2 + \cdots + x_n$. The theorem statement is trivial for $s = 0$ by the definition of a C-position. Assume for some $k > 0$ that the statement holds for all $s < k$. Consider a position $P = (x_1, x_2, \cdots, x_n)$ with $s = k$. Clearly any move will decrease s. Let $s_3(P) = (x_1)_T \boxplus (x_2)_T \boxplus \cdots \boxplus (x_n)_T$. We will prove and use the following lemma.

    Lemma 1. Any position P with $s_3(P) \neq 0$ can be moved to one with $s_3(P) = 0$ in at most two moves, while a position P with $s_3(P) = 0$ cannot be moved to another such position in at most two moves.

    Proof. For the first part of the lemma: Let $s_3(P) = \overline{c_{d-1} c_{d-2} \cdots c_1 c_0}_3$ in base 3 with c_{d-1} nonzero. If we can show that the lemma is true when the sizes of all piles have $\leq d$ digits in base 2, then in any other case we can ignore all but the last d digits of each pile and perform the same operation on these digits to get the same result. Let the largest 3 piles in P have sizes $a_1 > a_2 > a_3$. We are assuming that a_1 has at most d digits in base 2. Let $c_{d-1} = j$. We use strong induction on d to show that it is possible to perform the operation specified in the lemma with all of $a_1, a_2, \cdots, a_j$ changed. For $d = 0$, this is trivial because all the nonempty piles are 1s, so we remove the c_0 biggest piles and are done. Now assume it's true for all $d < d_0$ for some $d_0 > 0$, and consider $d = d_0$. Let $c_{d0-1} = j_0$. Then replace $a_1, a_2, \cdots, a_{j0}$ with $a^* = 2^{d_0 - 1} - 1$. This results in a position P' with $d = d' \leq d_0 - 1$ (since $c_{d_0 - 1}$ is now 0), so by induction the $c'_{d'-1}$ piles whose last d' digits form the largest base 2 integers possible can be used to do the operation demanded by the lemma. Since all digits of a^* in base 2 are 1, clearly $a_1, a_2, \cdots, a_j$, which have all been replaced by a^*, can be used from now on for any number of digits $d < d_0$ until more than j_0 digits are needed for some future digit, in which case we continue using them along with as many of the next largest integers as necessary. So the statement holds for $d = d_0$ as well, and by strong induction holds for all d. Now we prove the second part of the lemma. Assume the opposite, so there are piles a, b in P such that there exist nonnegative integers $a' < a$, $b' \le b$ with $a_T \boxplus b_T = s_3(P) \boxplus a'_T \boxplus b'_T = a'_T \boxplus b'_T$. Since the digits of a_T, b_T, a'_T, b'_T in base 3 are 0s and 1s, summing them pairwise cannot produce "'carry-overs"', so $a_T \boxplus b_T = a_T + b_T$ and $a'_T \boxplus b'_T = a'_T + b'_T$. But then we get $a_T + b_T = a'_T + b'_T$, which is impossible if $a' < a$ and $b' \le b$ (since it is simple to see, e.g. by expanding out the integers' base representations into polynomials of the base, that $a > a'$ implies $a_T > a'_T$, etc.). This completes the proof of the lemma. □

Now we continue with the main proof. If $s_3(P)$ is nonzero, then there exist two piles a, b that can be altered (with b possibly not changed) to produce a position P' with $s_3(P') = 0$. Then the player (without loss of generality, player 1) who is faced with position P can alter a. If this leaves a position P' with $s_3(P') = 0$ already, then P' is a C position by the inductive hypothesis, and P is an A-position. Otherwise, player 2 can alter b to leave a position P' with $s_3(P') = 0$, which is a C-position for player 2, so player 1 would be in an A-position, implying P is a B-position. In either case, P is not a C position if $s_3(P) \neq 0$.

If instead $s_3(P) = 0$, notice that altering any pile will result in a position P' with $s_3(P') \neq 0$ (by, e.g., the lemma). P' is then either an A-position or a B-position. However, by the lemma applied to P, P' cannot be moved to a position P'' with $s_3(P'') = 0$ in one move, and since by the inductive hypothesis such positions are the only C-positions, P' cannot be an A-position, so it must be a B-position, and P is a C-position as wanted. So the theorem holds for $s = k$ as well, and therefore for all s. □

29. Let $ABCD$ be a tetrahedron whose six side lengths are all integers, and let N denote the sum of these side lengths. There exists a point P inside $ABCD$ such that the feet from P onto the faces of the tetrahedron are the orthocenter of $\triangle ABC$, centroid of $\triangle BCD$, circumcenter of $\triangle CDA$, and orthocenter of $\triangle DAB$. If $CD = 3$ and $N < 100{,}000$, determine the maximum possible value of N.

Proposed by Sammy Luo and Evan Chen.

Answer. $\boxed{15000}$.

Solution. Let H, G, O, H' denote the orthocenter of $\triangle ABC$, centroid of $\triangle BCD$, circumcenter of $\triangle CDA$, and orthocenter of $\triangle DAB$, respectively.

The standard perpendicularity criterion applied to PH and BC and to PG and BC gives

$$BH^2 - CH^2 = BP^2 - CP^2 = BG^2 - CG^2$$

(so $\overline{BC} \perp \overline{GH}$), and similarly for the other five pairs of sides between $ABCD$ and $GOH'H$.

We can cancel a whole bunch of terms in these expressions, using for example the identities

$$BH^2 = 4R_{ABC}^2 - AC^2$$

and

$$BG^2 = \frac{2}{9}(BC^2 + CD^2 + DB^2) - \frac{1}{3}CD^2.$$

Due to the circumcenter we get

$$GC^2 - GD^2 = HA^2 - HC^2 = H'D^2 - H'A^2 = 0.$$

Hence we define $s = BC = BD = BA$. The remaining equations give

$$\frac{CD^2 - s^2}{3} = DA^2 - s^2 = AC^2 - s^2.$$

Letting $DA = AC = a$, we find $3a^2 - 2s^2 = 9$. This is a Pell-type equation, and the only solutions small enough are $(a, s) = (3, 3), (27, 33), (267, 327), (2643, 3237)$. So, using the largest one, the answer is $3 + 2 \cdot 2643 + 3 \cdot 3237 = 15000$. □

30. For a positive integer n, an n-*branch* B is an ordered tuple $(S_1, S_2, \dots, S_m)$ of nonempty sets (where m is any positive integer) satisfying $S_1 \subset S_2 \subset \cdots \subset S_m \subseteq \{1, 2, \dots, n\}$. An integer x is said to *appear* in B if it is an element of the last set S_m. Define an n-*plant* to be an (unordered) set of n-branches $\{B_1, B_2, \dots, B_k\}$, and call it *perfect* if each of 1, 2, ..., n appears in exactly one of its branches.

Let T_n be the number of distinct perfect n-plants (where $T_0 = 1$), and suppose that for some positive real number x we have the convergence

$$\ln\left(\sum_{n\ge 0} T_n \cdot \frac{(\ln x)^n}{n!}\right) = \frac{6}{29}.$$

If $x = \frac{m}{n}$ for relatively prime positive integers m and n, compute $m + n$.

Proposed by Yang Liu.

Answer. $\boxed{76}$.

Solution. This is a fairly transparent use of generating functions. The main claim is that

$$\ln \sum_{n\ge 0}\left(T_n \frac{x^n}{n!}\right) = \frac{e^x - 1}{2 - e^x}.$$

From this the answer is evidently just the solution to $\frac{a-1}{2-a} = \frac{6}{29}$, or $a = \frac{41}{35}$, giving $41 + 35 = 76$. Hence, we simply need to establish the above claim.

Let $S(n, k)$ denote the Stirling numbers of the second kind.

Lemma 1. We have

$$\sum_{n\ge 0}\left(x^n \cdot \sum_{a_1+2a_2+\dots na_n=n} \frac{1}{a_1!a_2!\dots a_n!}\right) = e^{\frac{x}{1-x}}.$$

where the a_i in the second sum are nonnegative integers.

Proof. Compute

$$\sum_{n\ge 0}\left(x^n \cdot \sum_{a_1+2a_2+\dots na_n} \frac{1}{a_1!a_2!\dots a_n!}\right) = \prod_{i\ge 1}\left(\sum_{j\ge 0}\frac{x^{ij}}{j!}\right) = \prod_{i\ge 1} e^{x^i} = e^{\frac{x}{1-x}}.$$ □

For simplicity later, let

$$A_k = \sum_{a_1+2a_2+\dots na_n} \frac{1}{a_1!a_2!\dots a_n!}.$$

Now, we directly count T_n. To do this, we do casework on how many vertices we have not including the root. Say we have k vertices, and say from under the root, we have i branches of length a_i. Then clearly, $a_1 + 2a_2 + \ldots + ka_k = k$. Clearly there are $k!S(n, k)$ ways to assign the numbers 1 to n to k vertices such that each vertex has at least one number assigned to it. But this overcounts slightly: in fact, we need to divide by $a_1!a_2!\dots a_n!$ in order to account for permutations of the branches of equal length. So

$$T_n = \sum_{k\ge 0} k!S(n, k)A_k$$

To finish,

$$\sum_{n\ge 0} T_n \cdot \frac{x^n}{n!} = \sum_{n\ge 0}\sum_{k\ge 0} k!S(n, k)A_k \cdot \frac{x^n}{n!} = \sum_{k\ge 0} k!A_k \sum_{n\ge 0} S(n, k) \cdot \frac{x^n}{n!} = \sum_{k\ge 0} A_k(e^x - 1)^k = e^{\frac{e^x-1}{2-e^x}},$$

by Lemma 1. Here we used the fact that $\sum_{n\ge 0} S(n, k) \cdot \frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}$. □