The Online Math Open Fall Contest
Official Solutions
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Acknowledgements

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1. Evalute
$$\sqrt{\binom{8}{2} + \binom{9}{2} + \binom{15}{2} + \binom{16}{2}}.$$

   Proposed by Evan Chen.

   Answer. $\boxed{17}$.

   Solution. The main observation is that
$$\binom{8}{2} + \binom{9}{2} = 8^2 \qquad \binom{15}{2} + \binom{16}{2} = 15^2$$
   so that the desired sum is $\sqrt{8^2 + 15^2} = 17$, a well-known Pythagorean triple. □

2. At a national math contest, students are being housed in single rooms and double rooms; it is known that 75% of the students are housed in double rooms. What percentage of the rooms occupied are double rooms?

   Proposed by Evan Chen.

   Answer. $\boxed{60}$.

   Solution. Assume there are k students in single rooms and $3k$ students in double rooms. The number of single rooms is k, and the number of double rooms is $\frac{3}{2}k$. So the answer is $\frac{\frac{3}{2}k}{\frac{3}{2}k+k} = \frac{3}{5}$, which is 60%. □

3. How many integers between 123 and 321 inclusive have exactly two digits that are 2?

   Proposed by Yannick Yao.

   Answer. $\boxed{18}$.

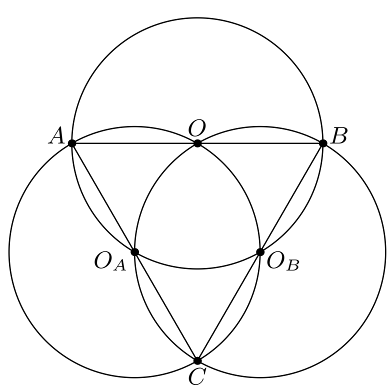
   Solution. These integers must take on the form $\overline{22n}$ or $\overline{2n2}$, where $n \neq 2$. Since there are 9 choices for n in each case, the answer is $2 \cdot 9 = 18$. □

4. Let ω be a circle with diameter AB and center O. We draw a circle ω_A through O and A, and another circle ω_B through O and B; the circles ω_A and ω_B intersect at a point C distinct from O. Assume that all three circles ω, ω_A, ω_B are congruent. If $CO = \sqrt{3}$, what is the perimeter of $\triangle ABC$?

   Proposed by Evan Chen.

   Answer. $\boxed{6}$.

   Solution. Let O_A be the center of ω_A and O_B the center of ω_B. Notice that O_A and O_B must lie on the same sides of line AB, since the assumptions of the problem implicitly tell us the circles are not tangent at O.



Moreover, by symmetry we have that $\overline{OC} \perp \overline{AB}$; so O_A and O_B are the midpoints of AC and BC. In particular, AOO_A and BOO_B are equilateral; finally we deduce ABC is equilateral too.

Since $OC = \sqrt{3}$, we find $AB = BC = CA = 2$, so the perimeter is 6. □

5. Merlin wants to buy a magical box, which happens to be an n-dimensional hypercube with side length 1 cm. The box needs to be large enough to fit his wand, which is 25.6 cm long. What is the minimal possible value of n?

   Proposed by Evan Chen.

   Answer. $\boxed{656}$.

   Solution. By the Pythagorean Theorem in n-dimensional space, the maximal length is given by the diagonal

$$\sqrt{\underbrace{(1-0)^2 + (1-0)^2 + \cdots + (1-0)^2}_{n \text{ times}}} = \sqrt{n}.$$

   This is the distance from $(0, \ldots, 0)$ to $(1, \ldots, 1)$ as points in $\mathbb{R}^n$. So we require $\sqrt{n} \ge 25.6 \iff n \ge 655.36$; the smallest integer n is $n = 656$. □

6. Farmer John has a (flexible) fence of length L and two straight walls that intersect at a corner perpendicular to each other. He knows that if he doesn't use any walls, he can enclose a maximum possible area of A_0, and when he uses one of the walls or both walls, he gets a maximum area of A_1 and A_2 respectively. If $n = \frac{A_1}{A_0} + \frac{A_2}{A_1}$, find $\lfloor 1000n \rfloor$.

   Proposed by Yannick Yao.

   Answer. $\boxed{4000}$.

   Solution. Since a circle is the shape that maximizes area given a fixed perimeter, with no walls the best area Farmer John can achieve is with a circle. With one wall, he makes a semicircle. This can be seen to be optimal by reflecting the region over the wall and considering their union. This must be a circle by above, so a semicircle is optimal. In the case with 2 walls, the region should be a quarter circle. Now it is easy to compute that $A_0 = \frac{L^2}{4\pi}, A_1 = \frac{L^2}{2\pi}, A_2 = \frac{L^2}{\pi}$.

   So $\frac{A_2}{A_1} + \frac{A_1}{A_0} = 4$, giving an answer of 4000. □

7. Define sequence $\{a_n\}$ as following: $a_0 = 0$, $a_1 = 1$, and $a_i = 2a_{i-1} - a_{i-2} + 2$ for all $i \ge 2$. Determine the value of a_{1000}.

   Proposed by Yannick Yao.

   Answer. $\boxed{1000000}$.

   Solution. We claim that in fact $a_n = n^2$ for every integer n. The proof is by induction on n with the base cases $n = 0$ and $n = 1$ given; for the inductive step, we observe that

$$2(i-1)^2 - (i-2)^2 + 2 = i^2$$

   as desired. Therefore $a_{1000} = 1000^2 = 1000000$. □

8. The two numbers 0 and 1 are initially written in a row on a chalkboard. Every minute thereafter, Denys writes the number $a + b$ between all pairs of consecutive numbers a, b on the board. How many odd numbers will be on the board after 10 such operations?

   Proposed by Michael Kural.

   Answer. $\boxed{683}$.

Solution. All numbers are integers at all points, so we will tacitly take modulo 2 everywhere. We claim that after k operations, the numbers on the board are

$$\underbrace{011011011\dots011}_{\frac{2^k-1}{3}\text{ blocks}}01$$

when k is even and

$$\underbrace{011011011\dots011}_{\frac{2^k+1}{3}\text{ blocks}}$$

when k is odd. (Note that in total, there are 2^k+1 numbers written.)

The proof of this observation is a direct induction on $k \ge 0$. Applying this to $k = 10$, we see the number of odd numbers is $\frac{2^{10}-1}{3} \cdot 2 + 1 = 683$. □

9. Let $s_1, s_2, \dots$ be an arithmetic progression of positive integers. Suppose that

$$s_{s_1} = x+2, \quad s_{s_2} = x^2+18, \quad \text{and} \quad s_{s_3} = 2x^2+18.$$

Determine the value of x.

Proposed by Evan Chen.

Answer. $\boxed{16}$.

Solution. The main observation is that s_{s_1}, s_{s_2}, s_{s_3} must be in arithmetic progression since s_1, s_2, and s_3 are. From this, we have that $x+2$, x^2+18 and $2x^2+18$ are in arithmetic progression, hence $2(x^2+18) = (2x^2+18)+(x+2)$ which gives $x = 16$ immediately.

In fact, the sequence in question is $s_n = 16n - 14$. □

10. For any positive integer n, define a function f by

$$f(n) = 2n + 1 - 2^{\lfloor \log_2 n \rfloor + 1}.$$

Let f^m denote the function f applied m times.. Determine the number of integers n between 1 and 65535 inclusive such that $f^n(n) = f^{2015}(2015)$.

Proposed by Yannick Yao.

Answer. $\boxed{8008}$.

Solution. By observing the base-2 expansion of the integer, we see that the function is equivalent to removing the frontmost nonzero digit (which is 1) and adding a 1 at the end. Thus $f^n(n) = 2^{s(n)} - 1$, where $s(n)$ is the sum of binary digits of n. Since $2015 = 11111011111_2$ has $s(2015) = 10$, Therefore it suffices to find the number of positive integers with at most 16 binary digits exactly 10 of which are 1. This is $\binom{16}{10} = 8008$. □

11. A trapezoid $ABCD$ lies on the xy-plane. The slopes of lines BC and AD are both $\frac{1}{3}$, and the slope of line AB is $-\frac{2}{3}$. Given that $AB = CD$ and $BC < AD$, the absolute value of the slope of line CD can be expressed as $\frac{m}{n}$, where m, n are two relatively prime positive integers. Find $100m + n$.

Proposed by Yannick Yao.

Answer. $\boxed{1706}$.

Solution. WLOG set $\vec{AB} = \langle -3, 2\rangle$, $\vec{DC} = \langle p, q\rangle$. Notice that $\vec{AB} + \vec{DC}$ is a vector that is perpendicular to BC or AD, which means that it has slope -3. Combined with the condition that $AB = CD$, we can set up a system of equation to solve for p, q:

$$\frac{q+2}{p-3} = -3$$

$$p^2 + q^2 = (-3)^2 + 2^2 = 13$$

This system can be reduced to the quadratic equation $5p^2 - 21p + 18 = 0$, which gives $p = 3$ or $p = \frac{6}{5}$. Since $p - 3 \neq 0$, we have $\langle p, q\rangle = \langle \frac{6}{5}, \frac{17}{5}\rangle$. Thus $\frac{m}{n} = \frac{q}{p} = \frac{17}{6}$ and our answer is 1706. □

12. Let a, b, c be the distinct roots of the polynomial $P(x) = x^3 - 10x^2 + x - 2015$. The cubic polynomial $Q(x)$ is monic and has distinct roots $bc - a^2$, $ca - b^2$, $ab - c^2$. What is the sum of the coefficients of Q?

   Proposed by Evan Chen.

   Answer. $\boxed{2015000}$.

   Solution. Considering the factorization of Q, we seek to compute $(1 - bc + a^2)(1 - ca + b^2)(1 - ab + c^2)$. Since $1 = ab + bc + ca$ by Vieta's Formulas, this rewrites as

   $$(a(a+b+c))(b(a+b+c))(c(a+b+c)) = abc(a+b+c)^3 = 2015000.$$ □

13. You live in an economy where all coins are of value $1/k$ for some positive integer k (i.e. $1, 1/2, 1/3, \ldots$). You just recently bought a coin exchanging machine, called the *Cape Town Machine*. For any integer $n > 1$, this machine can take in n of your coins of the same value, and return a coin of value equal to the sum of values of those coins (provided the coin returned is part of the economy). Given that the product of coins values that you have is 2015^{-1000}, what is the maximum number of times you can use the machine over all possible starting sets of coins?

   Proposed by Yang Liu.

   Answer. $\boxed{308}$.

   Solution. Note that inserting a composite number ab of coins into the machine is always nonoptimal. Indeed, you can use the machine more by splitting the coins into b groups of size a, inserting each of these groups separately, and then inserting all the new coins that you get.

   Let S denote the product of the denominators of the coins values. Note that when you insert n coins on value $\frac{1}{kn}$ into the machine to return a $\frac{1}{k}$, S divides by $k^{n-1}n^n$. At any stage of the process, the S is always a positive integer. Therefore, to optimally use the machine, we should set $k = 1$ at all times, and n as primes that divide 2015.

   Therefore, we care about how many times we can divide 2015^{1000} by $5^5, 13^{13}, 31^{31}$. Summing over these, we get $\lfloor \frac{2015}{5} \rfloor + \lfloor \frac{2015}{13} \rfloor + \lfloor \frac{2015}{31} \rfloor = 308$. This can be achieved by making all the coins have value either $1/5, 1/13$, or $1/31$. □

14. Let $a_1, a_2, \ldots, a_{2015}$ be a sequence of positive integers in $[1, 100]$. Call a nonempty contiguous subsequence of this sequence *good* if the product of the integers in it leaves a remainder of 1 when divided by 101. In other words, it is a pair of integers (x, y) such that $1 \le x \le y \le 2015$ and

   $$a_x a_{x+1} \ldots a_{y-1} a_y \equiv 1 \pmod{101}.$$

   Find the minimum possible number of good subsequences across all possible (a_i).

   Proposed by Yang Liu.

   Answer. $\boxed{19320}$.

Solution. Consider the prefix products, i.e. $p_i = a_1a_2 \ldots a_i$. Define $p_0 = 1$. Note that (x, y) is good iff $p_{x-1} \equiv p_y \pmod{101}$. Noe define s_i to be the nubmer of prefix products that evaluate to $i \pmod{p}$ for $0 \le i \le 100$. Since all elements of the sequence are in the region $[1, 100]$, $s_0 = 0$. Therefore, $\sum_{i=1}^{100} s_i = 2016$. Given the definition of s_i, the number of good subsequences can be expressed as $\sum_{i=1}^{100} \binom{s_i}{2}$.

Since the function $\binom{x}{2} = \frac{x(x-1)}{2}$ is convex, the above function is minimized when the s_i are all "as close together as possible". This can be rigorized by Karamata's Inequality. Given all this, the answer is

$$\sum_{i=1}^{100} \binom{s_i}{2} \ge 84\binom{20}{2} + 16\binom{21}{2} = 19320.$$

□

15. A regular 2015-simplex $\mathcal{P}$ has 2016 vertices in 2015-dimensional space such that the distances between every pair of vertices are equal. Let S be the set of points contained inside $\mathcal{P}$ that are closer to its center than any of its vertices. The ratio of the volume of S to the volume of $\mathcal{P}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m + n$ is divided by 1000.

Proposed by James Lin.

Answer. $\boxed{321}$.

Solution. I'll show that a specific vertex has a $\left(\frac{2015}{4032}\right)^{2015}$ chance of being closer to a particular vertex than the center. It is easier to consider the 2016-simplex in 2016 space, as the vertices are then just $A_1 = (1, 0, \cdots, 0), A_2 = (0, 1, 0, \cdots, 0), \cdots, A_{2016}(0, \cdots, 0, 1)$. The center O is then $\left(\frac{1}{2016}, \frac{1}{2016}, \cdots, \frac{1}{2016}\right)$. Consider the vertex A_1 of the simplex. Let M be the midpoint of OA_1, and let N be point where the line OA_1 hits the opposite face formed by the points $A_2, A_3, \cdots, A_{2016}$. N would be the center of those faces, so $N = \left(0, \frac{1}{2015}, \frac{1}{2015}, \cdots, \frac{1}{2015}\right)$. A direct computation shows that $\frac{A_1M}{A_1N} = \frac{2015}{4032}$.

Consider where the perpendicular bisector of A_1O cuts the simplex. This creates a simplex similar to the larger one, with volume ratio $\left(\frac{A_1M}{A_1N}\right)^{2015}$, as the simplex is 2015 dimensional. This shows that the probability of being closer to A_1 than to O is just as we claimed.

No point can be closer to two vertices than the center. This can be seen with the argument of taking the perpendicular bisector of A_1O. No two of these created simplices will intersect because $2 \cdot \frac{2015}{4032} < 1$. Therefore, our desired probability is then $1 - 2016\left(\frac{2015}{4032}\right)^{2015}$. When taking $\pmod{1000}$, you should not forget to divide out the 2016. □

16. Given a (nondegenerate) triangle ABC with positive integer angles (in degrees), construct squares BCD_1D_2, ACE_1E_2 outside the triangle. Given that D_1, D_2, E_1, E_2 all lie on a circle, how many ordered triples $(\angle A, \angle B, \angle C)$ are possible?

Proposed by Yang Liu.

Answer. $\boxed{223}$.

Solution. I claim that the four points will be concyclic if and only if $\angle A = \angle B$, or $C = 45^\circ$. Assuming that $D_2D_1E_1E_2$ is cyclic, the perpendiculars bisectors of D_1D_2 and E_1E_2 meet at the circumcenter of $D_2D_1E_1E_2$. But the perpendicular bisectors of D_1D_2 and E_1E_2 are just the perpendicular bisectors of BC, AC, so they meet at O, the circumcenter of ABC. In other words, the circumcenters of $D_2D_1E_1E_2$ and ABC coincide. From here, one can use trigonometry and angles to compute OD_1, OE_1 and set them equal, solving for the angles. This simplifies nicely and finishes the problem. A synthetic approach follows though.

Because $AC = CE_1, BC = CD_1, \angle ACD_1 = \angle BCE_1, \triangle ACD_1 \cong \triangle E_1CB$. This implies that $AD_1 = E_1B$. Along with the fact that $OB = OC, OD_1 = OE_1$, we get that $\triangle BOE_1 \cong \triangle COD_1$. At this point, we get two cases. Either BOE_1, COD_1 are similarly, or oppositely oriented. If they are oppositely oriented, then $\angle AOE_1 = \angle AOD_1 - \angle D_1OE_1 = \angle BOE_1 - \angle D_1OE_1 = \angle BOD_1$. Now because $OB = OC, OD_1 = OE_1, \triangle BOD_1 \cong \triangle AOE_1 \implies AE_1 = BD_1 \implies CA = CB$, which is the isosceles case.

Now assume that the triangles are similarly oriented. Now a spiral similarity centered at O sends BE_1 to AD_1. Therefore, $\angle D_1OE_1 = \angle BOA = 2\angle C$. Note that a spiral similarity centered at C sends BD_1 to E_1A. By the spiral similarity lemma, if we set $X = BE_1 \cap AD_1$, $\angle BXD_1 = \angle BCD_1 = 90^\circ \implies 90^\circ = \angle D_1XE_1 = \angle D_1OE_1 = 2\angle C \implies \angle C = 45^\circ$. This finishes the proof. Given the lemma, extracting the desired answer is easy.

The second paragraph can just be seen with solely angle chasing and similar triangles; I just wanted to present a more conceptual approach. □

17. Let $x_1 \ldots, x_{42}$ be real numbers such that $5x_{i+1} - x_i - 3x_ix_{i+1} = 1$ for each $1 \le i \le 42$, with $x_1 = x_{43}$. Find the product of all possible values for $x_1 + x_2 + \cdots + x_{42}$.

Proposed by Michael Ma.

Answer. $\boxed{588}$.

Solution. First we notice that we can rearrange the terms of the condition into $x_{n+1} = \frac{x_n+1}{-3x_n+5}$. So we let $f(x) = \frac{x+1}{-3x+5}$. Now $f(x_n) = x_{n+1}$. So we can see that $f^{(42)}(x_m) = x_m$, for every m.

If we set

$$A = \begin{pmatrix} 1 & 1 \\ -3 & 5 \end{pmatrix}$$

then the coefficients of $f^{(n)}(x)$ are the entries of A^n. Now to calculate A^{42} we need to diagonalize A. So diagonalizing A as PBP^{-1} we get $P = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$. Thus

$$A^{42} = \begin{pmatrix} 3 \times 2^{99} - 2^{199} & 2^{199} - 2^{99} \\ 3 \times 2^{99} - 3 \times 2^{199} & 3 \times 2^{199} - 2^{99} \end{pmatrix}.$$

Now substituting back into $f^{(42)}(x_m) = x_m$ we get that $3x_n^2 - 4x_n + 1 = 0$. So now we conclude that $x_m = 1$ or $x_m = \frac{1}{3}$ for every m. Also notice that $f(1) = 1$ and $f(\frac{1}{3}) = \frac{1}{3}$. So finishing we see that the two possibilities are 42 and 14. Multiplying we get $42 \times 14 = 588$. □

18. Given an integer n, an integer $1 \le a \le n$ is called *n-well* if

$$\left\lfloor \frac{n}{\lfloor n/a \rfloor} \right\rfloor = a.$$

Let $f(n)$ be the number of n-well numbers, for each integer $n \ge 1$. Compute $f(1)+f(2)+\ldots+f(9999)$.

Proposed by Ashwin Sah.

Answer. $\boxed{1318350}$.

Solution. Let $n = ba + r$, where $0 \le r < a$. Then a is *n-well* if

$$a = \left\lfloor \frac{ba + r}{b} \right\rfloor = a + \left\lfloor \frac{r}{b} \right\rfloor$$

or equivalently, if $r < b$. Thus it suffices to compute the number of triples (a, b, r) of positive integers such that $ab + r < 10000$ and $r < \min(a, b)$.

First, fix a and consider the number of triples with $a < b$. In this case, we obtain all numbers $ab + r$ with $r < a < b$. But every positive integer that is at least $a(a+1)$ can be written uniquely in this form by the division algorithm, so for each a we get $10000 - a(a+1)$ solutions.

The case $a > b$ is symmetric, so we now consider the case $a = b$. In this case, any number in the form $a^2 + r$ satisfies $a^2 + r < a^2 + a < (a+1)^2$, so the integers between 1 and 9999 which can be written in the form $a^2 + r$ can only be done so in one way. Thus the number of valid triples (a, a, r) is simply the number of pairs (a, r) with $1 \leq a \leq 99$ and $0 \leq r < a$, which is $1 + 2 + \cdots + 99$. So the final answer is

$$\sum_{k=1}^{99} 2(10000 - a^2 - a) + a = 1980000 - \frac{99 \cdot 100}{2} - 2 \cdot \frac{99 \cdot 100 \cdot 199}{6} = 1318350$$

□

19. For any set S, let $P(S)$ be its power set, the set of all of its subsets. Over all sets A of 2015 arbitrary finite sets, let N be the maximum possible number of ordered pairs (S, T) such that $S \in P(A), T \in P(P(A))$, $S \in T$, and $S \subseteq T$. (Note that by convention, a set may never contain itself.) Find the remainder when N is divided by 1000.

    Proposed by Ashwin Sah.

    Answer. $\boxed{872}$.

    Solution. Let $k = 2015$. We might as well add in extra elements to make $|A| = k$, since this can only increase the amount of ordered pairs in question.

    Now, $T \in P(P(A))$ means that $S \subseteq T \subseteq P(A)$ and $S \in P(A)$ means that $S \subseteq A$. Combining gives $S \subseteq P(A) \cap A$. Let $|P(A) \cap A| = x$.

    Then there are $\binom{x}{i}$ possibilities for S where $|S| = i$. Then T must contain $S, a_1, a_2, \ldots, a_i$, where $a_1, \ldots, a_i$ are the distinct elements of S, which must be distinct from S itself (since they are elements of A and thus are finitely defined, and since they are also elements of S). Then T has $2^k - i - 1$ other elements that it can include or not, for a total of $\binom{x}{i} 2^{2^k - i - 1}$ possibilities when $|S| = i$. Vary i to get $\sum_{i=0}^{x} \binom{x}{i} 2^{2^k - i - 1} = 2^{2^k - 1} \left(\frac{3}{2}\right)^x$. Now this is maximized when x is, and $x \leq k$ is clear. Furthermore, we can attain $x = k$ at $A = \{\{\}, \{\{\}\}, \{\{\{\}\}\}, \ldots\}$, where there are k nested curly braces in the last element, since each element of A is in this case also a subset of A. Then $N_k = 2^{2^k - k - 1} 3^k$. We can then find this $\pmod{1000}$ fairly easily.

    An easier to understand version of the optimal A is $A = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \ldots\}$, up to k terms, where $\varnothing = \{\}$ is the empty set. □

20. Amandine and Brennon play a turn-based game, with Amadine starting. On their turn, a player must select a positive integer which cannot be represented as a sum of nonnegative multiples of any of the previously selected numbers. For example, if $3, 5$ have been selected so far, only $1, 2, 4, 7$ are available to be picked; if only 3 has been selected so far, all numbers not divisible by three are eligible. A player loses immediately if they select the integer 1.

    Call a number n *feminist* if $\gcd(n, 6) = 1$ and if Amandine wins if she starts with n. Compute the sum of the *feminist* numbers less than 40.

    Proposed by Ashwin Sah.

    Answer. $\boxed{192}$.

    Solution. We claim that the *feminist* numbers are just the prime numbers greater than three. If we can show that each of those primes $p \geq 5$ is a winning position, then we are done - a feminist number n satisfies $\gcd(n, 6) = 1$ and obviously $n > 1$, so n has a prime divisor $q \geq 5$; if $n \neq q$ then after Amandine selects n then Brennon can select q, and it is as if Brennon started with the move q and thus he will win, and it is not a feminist number.

    Suppose Amandine starts with $p \geq 5$, a prime. Then say Brennon does a. Clearly $\gcd(a, p) = 1$, so now there are only finitely many guys that are left to be chosen, and by Chicken McNugget the biggest of these is $ap - a - p > 1$ since $p \geq 5$. We will do a Chomp-like nonconstructive proof.

Suppose that now Brennon will now win, regardless of what Amandine does. If Amandine does $ap - a - p$, then Brennon can do a winning move b. It is easy to see that $ap - a - p$ is actually a nonnegative combination of a, b, p; then Amandine should have done move b to begin with, and thus win! So actually Amandine wins with some number, we just don't know which.

Thus the answer is $5 + 7 + 11 + 13 + 17 + 19 + 23 + 29 + 31 + 37 = 192$. □

21. Toner Drum and Celery Hilton are both running for president. A total of 2015 people cast their vote, giving 60% to Toner Drum. Let N be the number of "representative" sets of the 2015 voters that could have been polled to correctly predict the winner of the election (i.e. more people in the set voted for Drum than Hilton). Compute the remainder when N is divided by 2017.

Proposed by Ashwin Sah.

Answer. $\boxed{605}$.

Solution. Suppose $m = \frac{2}{5} \cdot 2015$ people voted for Celery and $n = \frac{3}{5} \cdot 2015$ for Toner, where $m < n$. Then the amount of sets of voters that could be chosen with $d > 0$ more people voting for Toner than for Celery is $\sum_{i=0}^{m} \binom{m}{i}\binom{n}{d+i} = \sum_{i=0}^{m} \binom{m}{m-i}\binom{n}{d+i}$, where some of the terms at the end might be zero if $m + d > n$. By Vandermonde Convolution, this sum is just $\binom{m+n}{m+d}$. Now sum over the possible differences, which are $d = 1, 2, \dots, n$. We get $\sum_{i=m+1}^{i=m+n} \binom{m+n}{i}$.

Now $m+n = 2015$, so $\binom{m+n}{i} \equiv \binom{2015}{i} \equiv \frac{i+1}{2016}\binom{2016}{i+1} \equiv \left(\frac{i+1}{2016}\right)(-1)^{i+1} \equiv (i+1)(-1)^i \pmod{2017}$, using Wilson's Theorem.

So since $m = \frac{2}{5}(2015) = 806$, the sum is $-808+809-810+\ldots-2016 \equiv -808+604(-1) \equiv -1412 \equiv 605 \pmod{2017}$. □

22. Let $W = \ldots x_{-1}x_0x_1x_2\ldots$ be an infinite periodic word consisting of only the letters a and b. The minimal period of W is 2^{2016}. Say that a word U *appears* in W if there are indices $k \le \ell$ such that $U = x_kx_{k+1}\ldots x_\ell$. A word U is called *special* if Ua, Ub, aU, bU all appear in W. (The empty word is considered special) You are given that there are no special words of length greater than 2015.

Let N be the minimum possible number of special words. Find the remainder when N is divided by 1000.

Proposed by Yang Liu.

Answer. $\boxed{535}$.

Solution. We will prove that you can show that if a word U appears twice in a period of W, then it is part of a special word. To prove this, take a sequence that appears twice in a period of W. Call the occurrences S_1, S_2. Define an *extension* of a sequence S in W to mean adding the letter immediately following S in W to S. Now, extend S_1, S_2 to the right by one letter at a time until they become different. This must happen at some point by the definition of minimal period. Similarly, extend S_1, S_2 to the left in a similar way until the letter before them is no longer the same. After doing both these, we have constructed a special word which contains S_1. Therefore, by Pigeonhole, all words of length 2016 appear exactly once in a period of W, or else some word appears twice, and this extends to a special word of length greater than 2015. So for all words U of length at most 2015, aU, bU, Ua, Ub have length at most 2016, so they are the prefix of some word of length 2016, implying that they all appear in W. So all words of length at most 2015 are special! So our answer is $2^0 + 2^1 + 2^2 + \ldots + 2^{2015} = 2^{2016} - 1$.

A construction of this bound and a sequence which satisfies the condition can be given by the deBrujin sequences. You can find them easily on Wikipedia. □

23. Let $p = 2017$, a prime number. Let N be the number of ordered triples (a, b, c) of integers such that $1 \le a, b \le p(p-1)$ and $a^b - b^a = p \cdot c$. Find the remainder when N is divided by 1000000.

Proposed by Evan Chen and Ashwin Sah.

Answer. $\boxed{512256}$.

Solution. Let $p = 2017$. If $p|a$ or vice versa then $p|a, b$, and all of these work. This adds $(p-1)^2$ possibilities.

Now all numbers n from 1 to $p(p-1)$ not divisible by p can be uniquely determined by $0 \le f(n), g(n) \le p-2$, where $n \equiv g^{f(n)} \pmod p$, where g is a fixed primitive root, and where $n \equiv g(n) \pmod{p-1}$.

So now say $a \equiv g^w \pmod p, b \equiv x \pmod{p-1}, a \equiv y \pmod{p-1}, b \equiv g^z \pmod p$, where $0 \le w, x, y, z \le p-2$. Fixing these will uniquely determine a and b between 1 and $p(p-1)$. If $p|a^b - b^a$, the reduces to $p|g^{wx} - g^{yz} \Leftrightarrow p-1|wx - yz$, where $0 \le w, x, y, z \le p-2$. We'll find the number of four-tuples satisfying this in general. (As in, four tuples (w, x, y, z) such that $wx \equiv yz \pmod n$, for $0 \le w, x, y, z \le n-1$.)

First use the Chinese Remainder Theorem to reduce it to $wx \equiv yz \pmod{q^k}$ for primes q. Call the resulting answer in this case $f(q^k)$.

Then let $g(i)$ for $0 \le i \le k$ be the amount of solutions to $wx \equiv q^i \pmod{q^k}$. We can then easily see that the desired sum is $\sum_{i=0}^{k} g(i)^2 \phi(q^{k-i})$. This comes from the fact that there are $\phi(q^{k-i})$ numbers m from 1 to q^k such that $v_q(m) = i$. Squaring $g(i)$ comes from the number of ways for $wx \equiv q^i \pmod{q^k}$ times the number of ways for $yz \equiv q^i \pmod{q^k}$, which are both $g(i)$.

Now we'll turn to computing $g(i)$. If $0 \le i \le k-1$, there are $i+1$ ways to fix $v_q(w)$ and $v_q(x)$. Afterwards, there are exactly $\phi(q^k) = q^{k-1}(q-1)$ to multiply w, x by integers relatively prime to q^k to ensure that $wx \equiv q^i \pmod{q^k}$. Therefore, $g(i) = (i+1) \cdot \phi(q^k)$. Our total contribution in this case is

$$\sum_{i=0}^{k-1} (i+1)^2 \phi(q^k)^2 \cdot \phi(q^{k-i})$$

For $i = k$, if $v_p(w) = j \le k$, then we can choose w in $\phi(q^{k-j})$ ways, and then x such that $q^{k-j}|x$, in q^j ways. Therefore,

$$g(k) = \sum_{i=0}^{k} \phi(q^{k-j}) q^j.$$

The total contribution in this case is therefore

$$g(k)^2 \phi(1) = g(k)^2 = \left(\sum_{j=0}^{k} \phi(q^{k-j}) q^j \right)^2.$$

Given these above formulas, we can now compute the answer for $p-1 = 2016 = 2^5 \cdot 3^2 \cdot 7$. Straightforward computations give that $f(2^5) = 48640, f(3^2) = 945, f(7^1) = 385$. Therefore, our final answer is (adding back $(p-1)^2$ from the top) $f(2^5)f(3^2)f(7^1) + 2016^2 \equiv 512256 \pmod{10^6}$. $\square$

24. Let ABC be an acute triangle with incenter I; ray AI meets the circumcircle Ω of ABC at $M \neq A$. Suppose T lies on line BC such that $\angle MIT = 90^\circ$. Let K be the foot of the altitude from I to $\overline{TM}$. Given that $\sin B = \frac{55}{73}$ and $\sin C = \frac{77}{85}$, and $\frac{BK}{CK} = \frac{m}{n}$ in lowest terms, compute $m + n$.

Proposed by Evan Chen.

Answer. $\boxed{5702}$.

Solution. Let X be the major arc midpoint. Let ray XI meet the circumcircle of BIC (centered at M) again at J. Then $BICJ$ is harmonic, K is the midpoint of IJ and in particular lies on Ω.

Moreover, ray KX is the angle bisector of $\angle BKC$, so if we let L be the intersection of $\overline{IK}$ with $\overline{BC}$ we deduce

$$\frac{BK}{KC} = \frac{BT}{TC} = \left(\frac{BI}{IC}\right)^2 = \frac{\sin^2 C/2}{\sin^2 B/2}.$$

From the givens, we have $\cos B = \frac{48}{73}$, $\cos C = \frac{36}{85}$, and hence the requested ratio is

$$\frac{\frac{1}{2}(1-\frac{36}{85})}{\frac{1}{2}(1-\frac{48}{73})} = \frac{49\cdot 73}{85\cdot 25} = \frac{3577}{2125}.$$

Hence an answer of $3577 + 2125 = 5702$. □

25. Define $\|A - B\| = (x_A - x_B)^2 + (y_A - y_B)^2$ for every two points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ in the plane. Let S be the set of points (x, y) in the plane for which $x, y \in \{0, 1, \dots, 100\}$. Find the number of functions $f : S \to S$ such that $\|A - B\| \equiv \|f(A) - f(B)\| \pmod{101}$ for any $A, B \in S$.

Proposed by Victor Wang.

Answer. $\boxed{2040200}$.

Solution. We solve this problem with 101 replaced by an arbitrary prime $p \equiv 1 \pmod 4$.

Let $P(A, B)$ denote the proposition that $\|A - B\| \equiv \|f(A) - f(B)\| \pmod{101}$ for points in the plane A, B. Let 0 denote the point $(0, 0)$. Most of the dots below denote dot product.

First translate so that $f(0) = 0$ (we multiply the count by p^2 at the end).

Then considering $P(x, 0)$, $P(y, 0)$, and $P(x, y)$, and noting p is odd yields $f(x)\cdot f(y) \equiv x \cdot y \pmod p$ for all points x, y.

Now fix x, y. Then $f(x + y)\cdot f(t) \equiv x\cdot t + y\cdot t \equiv f(x)\cdot f(t) + f(y)\cdot f(t) \pmod p$; therefore, $[f(x + y) - f(x) - f(y)]\cdot f(t) \equiv 0 \pmod p$ for all points t.

Assume for the sake of contradiction that $f(x + y) - f(x) - f(y) = (a, b)$ with at least one of a, b nonzero. Then for all points t, we have $(a, b)\cdot f(t) \equiv 0 \pmod p$, or equivalently, for all t, there exists an integer $g(t)$ such that $f(t) = (-g(t)b, g(t)a) \pmod p$.

In particular, we have $u^2 + v^2 \equiv \|f(u, v)\|^2 \equiv g(u, v)^2(b^2 + a^2) \pmod p$ for all residues u, v. On the other hand, by Pigeonhole or direct calculations, we know that $u^2 + v^2$ can hit any residue $\pmod p$, because there are $\frac{p+1}{2}$ values that are achievable by u^2. Yet $g(u, v)^2(b^2+a^2)$ is a square times a constant $(b^2 + a^2)$, which covers at most $\frac{p+1}{2} < p$ values, contradiction.

Thus $f(x + y) \equiv f(x) + f(y)$ for all x, y. Let $f(1, 0) = (A, B)$ and $f(0, 1) = (C, D)$. Since $\|f(x)\|^2 = \|x\|^2$, we need $A^2 + B^2 \equiv C^2 + D^2 \equiv 1$. Therefore, $f(u, v) \equiv (Au + Cv, Bu + Dv)$. By linearity, we just need to check $u^2 + v^2 \equiv (Au + Cv)^2 + (Bu + Dv)^2$, which boils down to $0 \equiv 2(AC + BD)uv$ for all u, v.

Therefore our desired answer is simply p^2 times the number of solutions to the system $A^2 + B^2 \equiv C^2+D^2 \equiv 1$, $AC+BD \equiv 0$. We will use i to denote an integer such that $i^2 \equiv -1 \pmod p$. This integer exists as $p \equiv 1 \pmod 4$. We can parameterize $A+iB \equiv \alpha$, $A-iB \equiv \alpha^{-1}$, $C+iD \equiv \beta$, $C-iD \equiv \beta^{-1}$ for nonzero residues α, β. Now $AC+BD \equiv 0$ is equivalent to $(\alpha+\alpha^{-1})(\beta+\beta^{-1}) - (\alpha-\alpha^{-1})(\beta-\beta^{-1}) \equiv 0$, or $\alpha^2 + \beta^2 \equiv 0$. For each (nonzero) α, there are exactly 2 choices for β (as once again, -1 is a square $\pmod p$), so our final answer is $p^2[2(p - 1)] = 2p^2(p - 1)$.

Remark: Counting the number of solutions to $A^2 + B^2 \equiv C^2 + D^2 \equiv 1$, $AC + BD \equiv 0$ is essentially counting orthogonal 2×2 matrices over the finite field $\mathbb{F}_p$.

A more natural way to word the problem statement is that you are being asked to compute the number of isometries of the plane in $\mathbb{F}_p^2$. □

26. Let ABC be a triangle with $AB = 72, AC = 98, BC = 110$, and circumcircle Γ, and let M be the midpoint of arc BC not containing A on Γ. Let A' be the reflection of A over BC, and suppose MB meets AC at D, while MC meets AB at E. If MA' meets DE at F, find the distance from F to the center of Γ.

Proposed by Michael Kural.

Answer. $\boxed{231}$.

Solution 1. Let G be the intersection of BC and AM, O be the center of Γ, and F' the Miquel point of complete quadrilateral $ABCDEM$. Also, let ω be the circle centered at M passing through B and C. We claim that F' is the inverse of G with respect to Γ, and furthermore that $F' = F$.

First, Miquel's theorem on triangle AED with respect to F, C, and B implies that F' lies on DE. It is also well-known that $OF' \perp DE$.[1] For sake of completeness, this can be proved as follows:

F' is the spiral similarity center mapping MC to BA, so letting X, Y be the midpoints of MC, AB, we get that the same spiral similarity also maps X to Y. Thus $F'XYE$ is cyclic (since F' is the Miquel point of $MBXY$), but this is the circle with diameter OE, so $OF' \perp F'E$ as desired.

Now by Brokard's theorem, ED is the polar of G. This implies that O, G, F' are collinear, and furthermore that F' is the inverse of G with respect to Γ.

It then suffices to show that F' lies on MA'. Indeed, we claim that F' and A' are inverses with respect to ω. Note that

$$\angle MBA' = \angle A'BC - \angle MBC = \angle ABC - \angle MCB = \angle BEM = \angle BF'M$$

and similarly, $\angle MCA' = \angle MF'C$. A' is uniquely determined by these two angle conditions, so it must be the inverse of F'. Thus $F' = F$, as desired.

Finally, it's left to find $OF' = \frac{R^2}{OG}$, where R is the circumradius of $\triangle ABC$. Straightforward computation yields $s = 140$,

$$K = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{140 \cdot 30 \cdot 42 \cdot 68} = 840\sqrt{17}$$

and

$$R = \frac{abc}{4K} = \frac{72 \cdot 98 \cdot 110}{4 \cdot 840\sqrt{17}} = \frac{231}{\sqrt{17}}$$

where s, K, R refer to the semiperimeter, area, and circumradius of $\triangle ABC$ respectively. Additionally,

$$BG = \frac{72}{72+98} \cdot 110 = \frac{792}{17}.$$

Let N be the midpoint of BC, so $GN = \frac{143}{17}$. Then

$$OG = \sqrt{GN^2 + NO^2} = \sqrt{GN^2 + R^2 - AG^2} = \sqrt{\left(\frac{143}{17}\right)^2 + \frac{231^2}{17} - 55^2} = \frac{231}{17}$$

Giving us our final answer of

$$\frac{R^2}{OG} = \frac{231^2}{17} \cdot \frac{17}{231} = 231.$$

□

Solution 2. First, we prove a lemma.

Lemma 1 (Commutativity of Inversion)**.** Suppose (A), (B), (C) are circles centered at A, B, C, and denote by α, β, γ the inversions about these circles. Suppose that α switches (B) and (C). Then for any point P in the plane, $beta\alpha(P) = \alpha\gamma(P)$.[2]

Proof. Let Γ be the circle through P orthogonal to (A) and (B). Since inversion preserves orthogonality, Γ is orthogonal to (C) as well. Let (A) intersect Γ at A_1 and A_2, and define B_1, B_2, C_2, C_2 similarly. Note that AA_1, etc. are tangent to Γ, and that without loss of generality, A, B_1, C_1 and A, B_2, C_2 are collinear triples of points. Let AP meet Γ again at Q, CP meet Γ again at R, and AR meet Γ again at S. Now Q=$\alpha(P)$, $R = \gamma(P)$, and $S = \alpha\gamma(P)$, so it suffices to show that $S = \beta(Q)$, or that B, S, Q are collinear. But $(C_1, C_2; P, R)$ is harmonic and inversion preserves cross products, so after applying α, we get that $(B_1, B_2; S, Q)$ is harmonic as well. This implies B, S, Q are collinear. □

[1]See `http://yufeizhao.com/olympiad/cyclic_quad.pdf` for more details about this configuration.
[2]Credits to Victor Reis, from `http://www.artofproblemsolving.com/community/c6h595784p3536985`.

Now we can directly apply the above lemma on ω, Γ, and BC (as the image of Γ after inversion about ω is the line BC). Inversion with respect to a line is simply reflection across that line. Applying the lemma to point A implies that the inverse of A' with respect to ω is the inverse of G with respect to Γ (as G is the inverse of A with respect to ω). This easily leads us to the finish after showing that the inverse of G lies on ED as before. □

27. For integers $0 \le m, n \le 64$, let $\alpha(m, n)$ be the number of nonnegative integers k for which $\lfloor m/2^k \rfloor$ and $\lfloor n/2^k \rfloor$ are both odd integers. Consider a 65×65 matrix M whose (i, j)th entry (for $1 \le i, j \le 65$) is

$$(-1)^{\alpha(i-1,j-1)}.$$

Compute the unique integer $0 \le r < 1000$ such that $\det M \equiv r \pmod{1000}$.

Proposed by Evan Chen.

Answer. $\boxed{208}$.

Solution 1.

Lemma 2. Let A be a $n \times n$ invertible matrix, and let B be the $2n \times 2n$ block matrix

$$B = \begin{bmatrix} A & A \\ A & -A \end{bmatrix}.$$

Then $\det B = (-2)^n (\det A)^2$.

Proof. Perform row reductions which reduce A to a diagonal matrix on the first n rows of B, and then again on the second n rows of B. The determinant of B is preserved, and the new matrix is in the form

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 & \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n & 0 & 0 & \dots & \lambda_n \\ \lambda_1 & 0 & \dots & 0 & -\lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 & -\lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n & 0 & 0 & \dots & -\lambda_n \end{bmatrix}$$

where $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$. Subtracting each row k from each row $n + k$ yields the upper triangular matrix

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 & \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n & 0 & 0 & \dots & \lambda_n \\ 0 & 0 & \dots & 0 & -2\lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & -2\lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & -2\lambda_n \end{bmatrix}$$

Thus

$$\det B = \lambda_1 \lambda_2 \cdots \lambda_n (-2\lambda_1)(-2\lambda_2) \cdots (-2\lambda_n) = (-2)^n (\det A)^2$$

as desired. □

Now denote by M_n the matrix of side length n with entries as described in the problem. We claim that

$$M_{2^{n+1}} = \begin{bmatrix} M_{2^n} & M_{2^n} \\ M_{2^n} & -M_{2^n} \end{bmatrix}.$$

It suffices to show that for $0 \le i, j < 2^n$,

$$(-1)^{\alpha(i,j)} = (-1)^{\alpha(i+2^n,j)} = (-1)^{\alpha(i,j+2^n)} = -(-1)^{\alpha(i+2^n,j+2^n)}$$

But it's easy to see $\alpha(x, y)$ is the number of places for which x and y both have a 1 in their respectives binary representations. Thus

$$\alpha(i, j) = \alpha(i + 2^n, j) = \alpha(i, j + 2^n) = \alpha(i + 2^n, j + 2^n) - 1$$

implying the above.

We wish to compute $\det M_{65}$, and to do this, we first compute $\det M_{64}$. Note that $\det M_{2^0} = 1$, $\det M_{2^1} = -2$, and for $n \ge 1$,

$$\det M_{2^{n+1}} = (-2)^{2^n} (\det M_{2^n})^2 = 2^{2^n} (\det M_{2^n})^2$$

so is easy to prove by induction that for $n > 1$,

$$\det M_{2^n} = 2^{n \cdot 2^{n-1}}$$

Now consider M_{2^n+1}. After subtracting the first row from the last, the matrix is in the form

$$M_{2^n+1} = \begin{bmatrix} M_{2^n} & \vdots \\ 0 & -2 \end{bmatrix}.$$

where the bottom left 0 represents a row of 0s. Thus by cofactor expansion,

$$\det M_{2^n+1} = (-2) \det M_{2^n} = -2^{n \cdot 2^{n-1}+1}$$

Plugging in $n = 6$ yields a determinant of -2^{193}, which is easily computed to be $208 \pmod{1000}$. □

Solution 2. In M_{2^n}, consider any two different row vectors. We claim that they are orthogonal, i.e. that for any $0 \le i, j < 2^n$,

$$\sum_{k=0}^{2^n-1} (-1)^{\alpha(i,k)+\alpha(j,k)} = 0.$$

We prove this by induction. For $n = 1$, the base case, it is clear. Otherwise, let v_i and v_j be the row vectors corresponding to some i and j in $M_{2^{n+1}}$. Also, let $a_1, a_2, \cdots a_{2^n}$ and $b_1, b_2, \cdots b_{2^n}$ denote the first 2^n entries of v_i and v_j. By a similar argument to that of the first solution, we get that if $i < 2^n$, then

$$v_i = \langle a_1, a_2, \cdots a_{2^n}, a_1, a_2, \cdots a_{2^n} \rangle$$

and if $i \ge 2^n$, then

$$v_i = \langle a_1, a_2, \cdots a_{2^n}, -a_1, -a_2, \cdots - a_{2^n} \rangle$$

So if $i \not\equiv j \pmod{2^n}$, then the first and second halve of each of the two vectors are orthogonal, implying that v_i and v_j are orthogonal. If $j = i + 2^n$, then

$$v_i \cdot v_j = a_1^2 + \cdots + a_{2^n}^2 - a_1^2 - \cdots - a_{2^n}^2 = 0$$

so v_i and v_j are orthogonal in all cases.

Now since the matrix is symmetric, this implies that all entries in $M_{2^n}^2$ are 0 other than the entires on the diagonal. But the entries on the diagonal are clearly 2^n, so $(\det M_{2^n})^2 = \det M_{2^n}^2 = 2^{n \cdot 2^n}$. We can't actually determine the sign of $\det M_{2^n}$ using this solution approach, but we get that $\det M_{2^n} = \pm 2^{n \cdot 2^{n-1}}$, with the way to obtain the correct sign and how to finish described in the first solution. □

28. Let N be the number of 2015-tuples of (not necessarily distinct) subsets $(S_1, S_2, \dots, S_{2015})$ of $\{1, 2, \dots, 2015\}$ such that the number of permutations σ of $\{1, 2, \dots, 2015\}$ satisfying $\sigma(i) \in S_i$ for all $1 \le i \le 2015$ is odd. Let k_2, k_3 be the largest integers such that $2^{k_2}|N$ and $3^{k_3}|N$ respectively. Find $k_2 + k_3$.

   Proposed by Yang Liu.

   Answer. $\boxed{2030612}$.

   Solution. Call a permutation *good* if it satisfies the desired property.

   Consider each subset as a vector in $\{0,1\}^{2015}$, and write these vectors in a 2015×2015 matrix in order. Define the *permanent* of this matrix to be the unsigned determinant, i.e. for all permutations σ,

$$\sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

   By definition, this obviously counts the number of good permutations. On the other hand, the determinant is

$$\sum_{\sigma} (-1)^{\text{sgn}(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)},$$

   where sgn denotes the sign of the permutation.

   Now we work in $\mathbb{F}_2$ for the remainder of the solution. Note that permanent is congruent to determinant (mod 2), so we want a nonzero determinant in $\mathbb{F}_2$. This is equivalent to these 2015 vector to be linearly independent. To count this, choose the rows of the matrix one by one. The first row can clearly be picked in $2^{2015} - 1$ ways, since we can pick anything other than the 0-vector. When picking the $i+1$-st row, the span of the previous i rows has size 2^i by linear independence. So we can pick the $i+1$-st row in $2^{2015} - 2^i$ ways, that is, anything not in the span of the previous i rows. So the total number of n-tuples of subsets is

$$\prod_{i=0}^{2014} (2^{2015} - 2^i).$$

   Now we can see that $k_2 = 2029105$, and we can compute that $k_3 = 1507$ with some simple applications of the lifting the exponent lemma (LTE). So $k_2 + k_3 = 2030612$. □

29. Given vectors $v_1, \dots, v_n$ and the string $v_1 v_2 \dots v_n$, we consider valid expressions formed by inserting $n-1$ sets of balanced parentheses and $n-1$ binary products, such that every product is surrounded by a parentheses and is one of the following forms:

   - A "normal product" ab, which takes a pair of scalars and returns a scalar, or takes a scalar and vector (in any order) and returns a vector.
   - A "dot product" $a \cdot b$, which takes in two vectors and returns a scalar.
   - A "cross product" $a \times b$, which takes in two vectors and returns a vector.

   An example of a *valid* expression when $n = 5$ is $(((v_1 \cdot v_2)v_3) \cdot (v_4 \times v_5))$, whose final output is a scalar. An example of an *invalid* expression is $(((v_1 \times (v_2 \times v_3)) \times (v_4 \cdot v_5))$; even though every product is surrounded by parentheses, in the last step one tries to take the cross product of a vector and a scalar.

   Denote by T_n the number of valid expressions (with $T_1 = 1$), and let R_n denote the remainder when T_n is divided by 4. Compute $R_1 + R_2 + R_3 + \dots + R_{1{,}000{,}000}$.

   Proposed by Ashwin Sah.

   Answer. $\boxed{320}$.

Solution. So let S_n be the number of ways to insert $n-1$ pairs of parentheses among n vectors to get a scalar result and V_n with a vector result. So $S_1 = 0, V_1 = 1$.

A recursion argument gives that $S_n = \sum_{k=1}^{n-1} S_k S_{n-k} + \sum_{k=1}^{n-1} V_k V_{n-k}$ and $V_n = 2\sum_{k=1}^{n-1} S_k V_{n-k} + \sum_{k=1}^{n-1} V_k V_{n-k}$.

Define $S(x), V(x)$ to be the generating functions of $\{S_i\}, \{V_i\}$; so $S(x) = \sum_{i\geq 1} S_i x^i, V(x) = \sum_{i\geq 1} V_i x^i$. Then by the above recurrences, $S(x) = S(x)^2 + V(x)^2, V(x) - x = 2S(x)V(x) + V(x)^2$.

So $V(x) - x = 2S(x)V(x) + S(x) - S(x)^2$, and thus $V(x)(1 - 2S(x)) = x + S(x) - S(x)^2$.

Then $(1 - 2S(x))^2 S(x) = (1 - 2S(x))^2 S(x)^2 + (x + S(x) - S(x)^2)^2$.

For the below, note that $f(x)^2 = f(x^2)$ in $\mathbb{F}_2[[x]]$ (otherwise known as power series with coefficients $\pmod 2$) for a power series f.

Taking $\pmod 2$ gives $S(x) \equiv x^2 + S(x^4) \pmod 2$, so $S(x) \equiv x^2 + x^8 + x^{32} + x^{128} + \dots$ by recursively substituting $x \to x^4$.

Similarly, $\pmod 2$ gives $V(x) \equiv x + V(x^2) \pmod 2$, too, so $V(x) \equiv x + x^2 + x^4 + x^8 + \dots \pmod 2$.

Now we can move onto $\pmod 4$. Notice that stuff within squares can be taken modulo 2. In other words, knowing if $f(x) \equiv g(x) \pmod 2$, then $f(x)^2 \equiv g(x)^2 \pmod 4$.

So

$$S(x) \equiv (x^2 + x^8 + x^{32} + \dots)^2 + (x + x^2 + x^4 + x^8 + \dots)^2.$$

And

$$V(x) = x + V(x)^2 + 2S(x)V(x) \equiv$$
$$x + (x + x^2 + x^4 + x^8 + \dots)^2 + 2(x^2 + x^8 + x^{32} + \dots)(x + x^2 + x^4 + x^8 + \dots) \pmod 4.$$

Add the two to find

$$S(x) + V(x) \equiv$$
$$x + 2(x+x^2+x^4+x^8+\dots)^2 + (x^2+x^8+x^{32}+\dots)^2 + 2(x^2+x^8+x^{32}+\dots)(x+x^2+x^4+x^8+\dots) \pmod 4.$$

Then we can find that $R(x)$ is $1 \pmod 4$ for x powers of four, $2 \pmod 4$ for twice powers of four and numbers $n = 2^a + 2^b$ for $a \neq b$ and not both even. The rest are divisible by four.

Now $2^0, 2^1, \cdots, 2^{19}$ are the powers of two that are less than 1000000. From being powers of four and twice powers of 4, this contributes a sum of 30. There are $\binom{20}{2} - \binom{10}{2} = 145$ ways to choose two of the numbers from 0 to 19 to not be both even. These contribute a sum of $2 \cdot 145 = 290$. Summing both of these gives $30 + 290 = 320$. □

30. Ryan is learning number theory. He reads about the *Möbius function* $\mu : \mathbb{N} \to \mathbb{Z}$, defined by $\mu(1) = 1$ and

$$\mu(n) = -\sum_{\substack{d|n \\ d\neq n}} \mu(d)$$

for $n > 1$ (here $\mathbb{N}$ is the set of positive integers). However, Ryan doesn't like negative numbers, so he invents his own function: the *dubious function* $\delta : \mathbb{N} \to \mathbb{N}$, defined by the relations $\delta(1) = 1$ and

$$\delta(n) = \sum_{\substack{d|n \\ d\neq n}} \delta(d)$$

for $n > 1$. Help Ryan determine the value of $1000p + q$, where p, q are relatively prime positive integers satisfying

$$\frac{p}{q} = \sum_{k=0}^{\infty} \frac{\delta(15^k)}{15^k}.$$

Proposed by Michael Kural.

Answer. $\boxed{11007}$.

Solution 1. Throughout this solution, $[x^i]$ denotes the coefficient of x^i in some power series of x. Let $f(i,j) = \delta(3^i 5^j)$, and let

$$F(x,y) = \sum_{i,j\ge 0} f(i,j) x^i y^j.$$

Note that for $(i,j) \ne (0,0)$,

$$2f(i,j) = 2\delta(3^i 5^j) = \sum_{\substack{0\le k\le i\\ 0\le l\le j}} \delta(3^k 5^l) = \sum_{\substack{0\le k\le i\\ 0\le l\le j}} f(k,l)$$

Let

$$g(i,j) = \sum_{\substack{0\le k\le i\\ 0\le l\le j}} f(k,l)$$

Then

$$\frac{F(x,y)}{(1-x)(1-y)} = \sum_{i,j\ge 0} g(i,j)x^i y^j = -1 + \sum_{i,j\ge 0} 2f(i,j)x^i y^j = 2F(x,y) - 1$$

implying after rearrangement that

$$F(x,y) = \frac{1}{2}\left(1 + \frac{1}{1-2x-2y+2xy}\right).$$

Now it simply suffices to find the generating function for the *diagonal* of F, or the coefficients of its $x^i y^i$ terms.

If $G(x,y)$ is a formal power series with $G(0,0) = 0$, we can write

$$\frac{1}{1-G(x,y)} = 1 + G(x,y) + G(x,y)^2 + G(x,y)^3 + \cdots$$

so if we let $F_1(x,y) = \frac{1}{1-2x-2y+2xy}$ for simpification purposes, then

$$\begin{aligned} F_1(x,y) &= \left(\frac{1}{1+2xy}\right)\left(\frac{1}{1-\dfrac{2(x+y)}{1+2xy}}\right) \\ &= \left(\frac{1}{1+2xy}\right)\left(1 + \left(\frac{2(x+y)}{1+2xy}\right) + \left(\frac{2(x+y)}{1+2xy}\right)^2 + \cdots\right) \end{aligned}$$

Now it suffices to find $D(z) = \sum_{i\ge 0}([x^i][y^i]F_1)z^i$ (indeed, our final answer will be $\frac{1}{2}(1 + D(\frac{1}{15}))$. Note that as $1+2xy$ only has equal coefficients in x, y, we can disregard any terms in the second expression in the form $cx^i y^j$ for $i \ne j$ (why?).

If n is odd, then $(x+y)^n$ is homogeneous of odd degree, and thus all of its terms can be disregarded in computing $D(z)$.

If $n = 2k$ is even, then $(x+y)^n$ has exactly one symmetric term (a term in the form $cx^i y^i$), namely $\binom{2k}{k}x^k y^k$.

From these, and the fact that a term in the form $cx^i y^j$ for $i \ne j$ cannot have any symmetric terms when multiplied by a power series of xy, it follows that

$$D(z) = \left(\frac{1}{1+2z}\right)\left(\sum_{k\ge 0}\binom{2k}{k}\frac{(4z)^k}{(1+2z)^{2k}}\right)$$

It is well-known (and can be derived by the Binomial Theorem and the identity $\binom{-\frac{1}{2}}{k} = (-4)^n \binom{2k}{k}$) that

$$\frac{1}{\sqrt{1-4s}} = \sum_{k\geq 0} \binom{2k}{k} s^k$$

so

$$D(z) = \left(\frac{1}{1+2z}\right)\left(\frac{1}{\sqrt{1-\frac{16z}{(1+2z)^2}}}\right) = \frac{1}{\sqrt{1-12z+4z^2}}$$

Then directly plugging in $z = \frac{1}{15}$ yields our final fraction

$$\frac{1}{2}(1 + D(\frac{1}{15})) = \frac{11}{7}$$

and final answer $1000 \cdot 11 + 7 = 11007$ $\square$

Comment. There are several alternate solutions which use a similar approach. A more direct, but slightly less rigorous approach to finding $D(\frac{1}{15})$ is to evaluate the constant term of

$$F_1(x, \frac{1}{15x}) = \frac{1}{1 - 2x - \frac{2}{15x} + \frac{2}{15}}$$

considered as a doubly infinite formal series in x. The issue here is that in general, we can't use a formal Laurent series (series with negative powers of x) unless the negative powers are bounded below, but it will work for our purposes for extracting the answer. Indeed, simply expanding

$$\frac{15}{17}\left(\frac{1}{1-\frac{30x}{17}-\frac{2}{17x}}\right) = \frac{15}{17}\sum_{k\geq 0}\left(\frac{30x}{17}+\frac{2}{17x}\right)^k$$

and extracting the constant terms as before gives the correct answer.

If we instead use the *function* $F_1(x, \frac{1}{15x})$ on $\mathbb{C}$, we can use a technique from complex analysis, as opposed to combinatorial expansion. To find the constant term of an expansion of $F_1(x, \frac{1}{15x})$, it suffices to find the contour integral

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{x} F_1(x, \frac{1}{15x}) dx$$

where γ is a "small" counterclockwise circular path around 0. This is equal to the residue at the smaller of the two poles of $\frac{1}{x}F_1(x, \frac{1}{15x})$; formally, some care must be taken to show essentially how small the circle is, and thus which poles are contained within it. (The answer for the curious is that such a pole must of $\frac{1}{x}F_1(x, \frac{z}{x})$ must approach 0 as z approaches 0, where z is currently taken as $\frac{1}{15}$.)

So note that

$$\frac{1}{x}F_1(x, \frac{15}{x}) = \frac{1}{x - 2x^2 - \frac{2}{15} + \frac{2x}{15}} = -\frac{1}{2}\frac{1}{(x-\frac{1}{6})(x-\frac{2}{5})}$$

which has residue $-\dfrac{1}{2(\frac{1}{6}-\frac{2}{5})} = \dfrac{15}{7}$ at $\frac{1}{6}$, yielding $D(\frac{1}{15}) = \frac{15}{7}$ as before.

Finally, using the factorization of the denominator above, there is simpler way to determine the coefficient of $\frac{1}{x}$, but again formally care would have to be taken regarding what the "correct" expansion in x is. Note that

$$-\frac{1}{2}\frac{1}{(x-\frac{1}{6})(x-\frac{2}{5})} = \frac{15}{7}\left(\frac{1}{x-\frac{1}{6}} - \frac{1}{x-\frac{2}{5}}\right)$$

but this cannot be expanded directly as a normal power series in x to determine the coefficient of $\frac{1}{x}$ here. Instead, for similar subtle reasons as above regarding which poles to take the residues at, we should expand this as

$$\frac{15}{7}\left(\frac{1}{x}\frac{1}{1-\frac{1}{6x}} + \frac{5}{2}\frac{1}{1-\frac{5x}{2}}\right) = \frac{15}{7}\left(\sum_{k\leq -1}\left(\frac{1}{6}\right)^{-1-k} x^k\right) + \frac{15}{7}\left(\sum_{k\geq 0}\left(\frac{5}{2}\right)^{k+1} x^k\right)$$

which has $\frac{15}{7}$ as the coefficient of $\frac{1}{x}$.

For more information regarding determining the diagonal of a generating function, see *Enumerative Combinatorics Volume 2* by Richard Stanley.

Solution 2. The following solution is due to Kevin Ren. Throughout this solution, the binomial coefficient $\binom{a}{b}$ is considered to be 0 if $a < 0$, $b < 0$,$b > a$,

Note that because the denominator of $F(x, y)$ from solution 1 is $1 - 2x - 2y + 2xy$, the recursion

$$f(i, j) = 2f(i - 1, j) + 2f(i, j - 1) - 2f(i - 1, j - 1)$$

holds for $i, j > 0$ and $(i, j) \neq (1, 1)$. This can also be derived directly from the recursive definition of $f(i, j)$. We claim by induction that for $(k, l) \neq (0, 0)$,

$$f(k, \ell) = \sum_{i \geq 0} \binom{k}{i}\binom{\ell}{i} 2^{k+\ell-i-1}$$

The base cases $f(k, 0) = f(0, k) = 2^{k-1}, k > 0$ and $f(1, 1) = 3$ are easily established. The inductive step follows as

$$\begin{aligned} f(k, l) &= 2\sum_{i \geq 0} 2^{k+\ell-i-3}\left(2\binom{k-1}{i}\binom{\ell}{i} + 2\binom{k}{i}\binom{\ell-1}{i} - \binom{k-1}{i}\binom{\ell-1}{i}\right) \\ &= \sum_{i \geq 0} 2^{k+\ell-i-1}\left(\binom{k-1}{i}\binom{\ell}{i} + \binom{k}{i}\binom{\ell-1}{i} - \binom{k-1}{i}\binom{\ell-1}{i} + \binom{k-1}{i-1}\binom{\ell-1}{i-1}\right) \\ &= \sum_{i \geq 0} 2^{k+\ell-i-1}\binom{k}{i}\binom{\ell}{i} \end{aligned}$$

where the second equality follows from

$$\sum_{i \geq 0} 2^{k+\ell-(i-1)-1}\left(\binom{k-1}{i-1}\binom{\ell-1}{i-1}\right) = \sum_{i \geq 0} 2^{k+\ell-i-1}\left(2\binom{k-1}{i}\binom{\ell-1}{i}\right)$$

and the third follows from the identity

$$\begin{aligned} \binom{k-1}{i-1}\binom{\ell-1}{i-1} &= \left(\binom{k}{i} - \binom{k}{i-1}\right)\left(\binom{\ell}{i} - \binom{\ell}{i-1}\right) \\ &= \binom{k}{i}\binom{\ell}{i} - \binom{k-1}{i}\binom{\ell}{i} - \binom{k}{i}\binom{\ell-1}{i} + \binom{k-1}{i}\binom{\ell-1}{i}. \end{aligned}$$

Thus now, to compute $D(z)$ as in the first solution, it suffices to derive an expression for

$$D(z) = \sum_{k \geq 0} z^k \sum_{i \geq 0} \binom{k}{i}^2 2^{2k-i}$$

Also note that

$$\sum_{i \geq 0} \binom{k}{i}^2 2^{2k-i} = \sum_{i \geq 0} \binom{k}{k-i}^2 2^{k+i} = \sum_{i \geq 0} \binom{k}{i}^2 2^{k+i}.$$

so we can derive

$$\begin{aligned}
D(z) &= \sum_{k\ge 0}(2z)^k \sum_{i\ge 0}\binom{k}{i}2^i \\
&= \sum_{k\ge 0}(2z)^k [t^k]((1+2t)(1+t))^k \\
&= \sum_{k\ge 0}(2z)^k [t^k]((1+3t+2t^2)^k \\
&= \sum_{k\ge 0}\sum_{n\ge 0}(2z)^k [t^k]\binom{k}{n}(3t+2t^2)^n \\
&= \sum_{m\ge 0}\sum_{n\ge m}(2z)^{m+n}[t^m]\binom{m+n}{n}(3+2t)^n \\
&= \sum_{m\ge 0}\sum_{n\ge m}(2z)^{m+n}\binom{m+n}{n}\binom{n}{m}2^m 3^{n-m} \\
&= \sum_{m\ge 0}\sum_{n\ge 0}\left(\frac{4z}{3}\right)^m (6z)^n \binom{2m}{m}\binom{m+n}{2m} \\
&= \sum_{m\ge 0}\left(\frac{4z}{3}\right)^m \binom{2m}{m}\left(\frac{(6z)^m}{(1-6z)^{2m+1}}\right) \\
&= \frac{1}{1-6z}\frac{1}{\sqrt{1-\frac{32z^2}{(1-6z)^2}}} \\
&= \frac{1}{\sqrt{1-12z+4z^2}}
\end{aligned}$$

and we can finish as before. □