The Online Math Open Fall Contest
Official Solutions
October 17 - 28, 2014

Online Math Open

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1. Carl has a rectangle whose side lengths are positive integers. This rectangle has the property that when he increases the width by 1 unit and decreases the length by 1 unit, the area increases by x square units. What is the smallest possible positive value of x?

   Proposed by Ray Li.

   Answer. $\boxed{1}$.

   Solution. If the intial length and width are a and b, respectively, then $x = (a-1)(b+1) - ab = ab - b + a - 1 - ab = -b + a - 1$, so choosing $b = 1, a = 3$ yields $x = 1$, the minimum possible positive integer. □

2. Suppose (a_n), (b_n), (c_n) are arithmetic progressions. Given that $a_1 + b_1 + c_1 = 0$ and $a_2 + b_2 + c_2 = 1$, compute $a_{2014} + b_{2014} + c_{2014}$.

   Proposed by Evan Chen.

   Answer. $\boxed{2013}$.

   Solution. Let $s_n = a_n + b_n + c_n$. We observe that s_n is also an arithmetic progression. From $s_1 = 0$ and $s_2 = 1$, we get that $s_n = n - 1$, so $s_{2014} = 2013$. □

3. Let $B = (20, 14)$ and $C = (18, 0)$ be two points in the plane. For every line ℓ passing through B, we color red the foot of the perpendicular from C to ℓ. The set of red points enclose a bounded region of area $\mathcal{A}$. Find $\lfloor \mathcal{A} \rfloor$ (that is, find the greatest integer not exceeding $\mathcal{A}$).

   Proposed by Yang Liu.

   Answer. $\boxed{157}$.

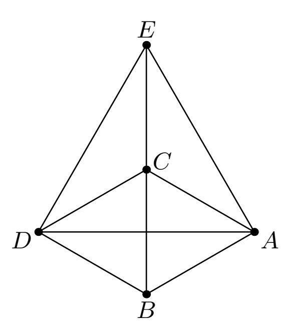
   Solution. For any possible red point F, $\angle BFC = \frac{\pi}{2}$, so F lies on the circle with diameter BC. Conversely, any point on this circle can be formed by some projection onto a line through B, so the set of red points is simply this circle. It has diameter $\sqrt{2^2 + 14^2}$, so its area is

$$\frac{1}{4}\left(2^2 + 14^2\right)\pi = 50\pi.$$ □

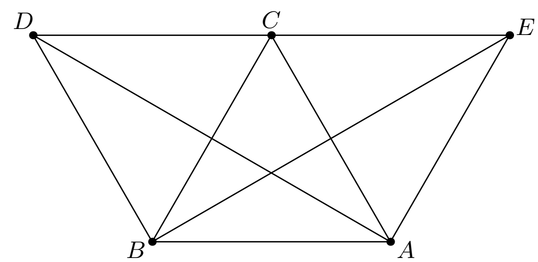
4. A crazy physicist has discovered a new particle called an emon. He starts with two emons in the plane, situated a distance 1 from each other. He also has a crazy machine which can take any two emons and create a third one in the plane such that the three emons lie at the vertices of an equilateral triangle. After he has five total emons, let P be the product of the $\binom{5}{2} = 10$ distances between the 10 pairs of emons. Find the greatest possible value of P^2.

   Proposed by Yang Liu.

   Solution. When we have 3 emons, they form an equilateral triangle. Then we the physicist adds a fourth emon, they form a rhombus with angles $60°, 120°$. For adding the fifth emon we have 2 cases: using the long diagonal of the rhombus, or using a side of the rhombus.



In the first case (using the long diagonal), we get that the ten lengths are: $1, 1, 1, 1, 1, 1, \sqrt{3}, \sqrt{3}, \sqrt{3}, 2$, so $P^2 = 108$.



In the second case (using a side of the rhombus), we get that the lengths are $1, 1, 1, 1, 1, 1, 1, \sqrt{3}, \sqrt{3}, 2$, so $P^2 = 36$. So our maximum value is 108. $\square$

5. A crazy physicist has discovered a new particle called an omon. He has a machine, which takes two omons of mass a and b and entangles them; this process destroys the omon with mass a, preserves the one with mass b, and creates a new omon whose mass is $\frac{1}{2}(a+b)$. The physicist can then repeat the process with the two resulting omons, choosing which omon to destroy at every step. The physicist initially has two omons whose masses are distinct positive integers less than 1000. What is the maximum possible number of times he can use his machine without producing an omon whose mass is not an integer?

   Proposed by Michael Kural.

   Answer. $\boxed{9}$.

   Solution. Consider the difference in the mass between the two particles. At each step, it becomes cut in half, so in order for it to remain an integer at each step, we want the intial difference to be divisible by the greatest power of 2 possible. This greatest possible power of 2 dividing the difference is clearly $512 = 2^9$, so the maximal number of times he can use his machine is 9, which can be obtained if we start with omons of mass 1 and 513 and arbitrarily destroy an omon at each step. $\square$

6. For an olympiad geometry problem, Tina wants to draw an acute triangle whose angles each measure a multiple of 10°. She doesn't want her triangle to have any special properties, so none of the angles can measure 30° or 60°, and the triangle should definitely not be isosceles.

   How many different triangles can Tina draw? (Similar triangles are considered the same.)

   Proposed by Evan Chen.

   Answer. $\boxed{0}$.

   Solution. Suppose the triangle has angles $10a < 10b < 10c < 90$ in degrees. Then $a < b < c < 9$ and $a + b + c = 18$. We now consider several cases.

   - If $c = 8$, we have $a + b = 10$ and $a < b \leq 7$. This gives $(a, b) = (3, 7)$ and $(a, b) = (4, 6)$, neither of which work.
   - If $c = 7$, we have $a + b = 11$ and $a < b \leq 6$. The only possibility here is $(a, b) = (5, 6)$, which also fails.
   - If $c \leq 6$, then $a, b < 6$, so $a + b + c < 18$. Hence no solutions can occur with $c \leq 6$.

Hence, Tina unfortunately cannot draw any triangles and the answer is 0. $\square$

7. Define the function $f(x, y, z)$ by

$$f(x, y, z) = x^{y^z} - x^{z^y} + y^{z^x} - y^{x^z} + z^{x^y}.$$

Evaluate $f(1, 2, 3) + f(1, 3, 2) + f(2, 1, 3) + f(2, 3, 1) + f(3, 1, 2) + f(3, 2, 1)$.

Proposed by Robin Park.

Answer. $\boxed{24}$.

Solution. Let $g(x, y, z) = f(x, y, z) - z^{y^x}$. By symmetry, we have that

$$g(1, 2, 3) + g(1, 3, 2) + g(2, 1, 3) + g(2, 3, 1) + g(3, 1, 2) + g(3, 2, 1) = 0,$$

and so

$$\begin{aligned} & f(1, 2, 3) + f(1, 3, 2) + f(2, 1, 3) + f(2, 3, 1) + f(3, 1, 2) + f(3, 2, 1) \\ = {} & g(1, 2, 3) + g(1, 3, 2) + g(2, 1, 3) + g(2, 3, 1) + g(3, 1, 2) + g(3, 2, 1) \\ + {} & (1^{2^3} + 1^{3^2} + 2^{1^3} + 2^{3^1} + 3^{1^2} + 3^{2^1}) = 24. \end{aligned}$$

□

8. Let a and b be randomly selected three-digit integers and suppose $a > b$. We say that a is *clearly bigger* than b if each digit of a is larger than the corresponding digit of b. If the probability that a is clearly bigger than b is $\frac{m}{n}$, where m and n are relatively prime integers, compute $m + n$.

Proposed by Evan Chen.

Answer. $\boxed{1061}$.

Solution. Compute

$$\frac{\binom{9}{2} \cdot \binom{10}{2}^2}{\binom{900}{2}} = \frac{36 \cdot 45^2}{450 \cdot 899} = \frac{162}{899}.$$

The numerator represents the number of ways to select a pair of hundreds digits, tens digits, and unit digits in a pair of clearly bigger numbers. The denominator represents the total number of pairs $a > b$ of three-digit numbers. Hence the answer is $162 + 899 = 1061$. □

9. Let $N = 2014! + 2015! + 2016! + \cdots + 9999!$. How many zeros are at the end of the decimal representation of N?

Proposed by Evan Chen.

Answer. $\boxed{501}$.

Solution. Let $\nu_p(n)$ denote the exponent of p in the prime factorization of n.We seek $\min(\nu_2(N), \nu_5(N))$. We can see that

$$\frac{N}{2014!} = 1 + 2015 + 2015 \cdot 2016 + 2015 \cdot 2016 \cdot 2017.$$

Hence we see that $\frac{N}{2014!}$ is an integer not divisible by 5. So $\nu_5(N) = \nu_5(2014!)$. By Legendre's Formula[1], this is $\frac{2014-10}{4} = 501$ (observe that $2014 = 31024_5$). And it is easy to check that $\nu_2(2014!) > 501$, so $\nu_2(N) > 501$ as well. Hence the answer is 501.

□

[1] See `http://www.aops.com/Wiki/index.php/Legendre's_Formula`, for example.

10. Find the sum of the decimal digits of

$$\left\lfloor \frac{51525354555657\dots979899}{50} \right\rfloor.$$

Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.

Proposed by Evan Chen.

Answer. $\boxed{457}$.

Solution. It's not hard to check that the resulting quotient is

$$N = 0103\dots97$$

(where we have adding a leading 0). If we consider N' which is N with 99 appended to the right, then we obtain a 100-digit number for which the average of the odd-indexed digits is 4.5 and the average of the even-indexed digits is 5. So, the sum of the digits of N' is $9.5 \cdot 50 = 475$.

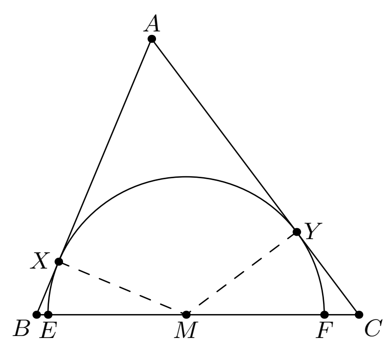
0Hence the final answer is $475 - 18 = 457$. □

11. Given a triangle ABC, consider the semicircle with diameter $\overline{EF}$ on $\overline{BC}$ tangent to $\overline{AB}$ and $\overline{AC}$. If $BE = 1$, $EF = 24$, and $FC = 3$, find the perimeter of $\triangle ABC$.

Proposed by Ray Li.

Answer. $\boxed{84}$.

Solution. Let M be the midpoint of EF and let the semicircle be tangent to $\overline{AB}$, $\overline{AC}$ at X, Y.



It is easy to see that $BM = 13$, so $BX = 5$. Similarly, $CM = 15$, so $CY = 9$. Now let $AX = AY = k$. Thus $AB = k + 5$ and $AC = k + 9$, and then $AM^2 = k^2 + 12^2$. Finally, $BC = 28$. Apply Stewart's Theorem to obtain

$$\left(k^2 + 12^2 + 13 \cdot 15\right) \cdot 28 = (k+5)^2 \cdot 15 + (k+9)^2 \cdot 13.$$

Solving, we find that $k = 21$. So $AB + AC + BC = 2k + 14 + 28 = 84$. □

12. Let a, b, c be positive real numbers for which

$$\frac{5}{a} = b + c, \quad \frac{10}{b} = c + a, \quad \text{and} \quad \frac{13}{c} = a + b.$$

If $a + b + c = \frac{m}{n}$ for relatively prime positive integers m and n, compute $m + n$.

Proposed by Evan Chen.

Answer. $\boxed{55}$.

Solution. We get $ab+ac=5$, $bc+ba=10$, $ca+cb=13$, so $ab+bc+ca=\frac{5+10+13}{2}=14$. Therefore, we have $ab=1$, $bc=9$, $ca=4$. Thus $abc=\sqrt{1\cdot 4\cdot 9}=6$ and

$$a+b+c=\frac{6}{1}+\frac{6}{4}+\frac{6}{9}=\frac{49}{6}.$$

So the answer is $49+6=55$. □

13. Two ducks, Wat and Q, are taking a math test with 1022 other ducklings. The test has 30 questions, and the nth question is worth n points. The ducks work independently on the test. Wat gets the nth problem correct with probability $\frac{1}{n^2}$ while Q gets the nth problem correct with probability $\frac{1}{n+1}$. Unfortunately, the remaining ducklings each answer all 30 questions incorrectly.

    Just before turning in their test, the ducks and ducklings decide to share answers! On any question which Wat and Q have the same answer, the ducklings change their answers to agree with them. After this process, what is the expected value of the sum of all 1024 scores?

    Proposed by Evan Chen.

    Answer. $\boxed{1020}$.

    Solution. By linearity of expectation, it suffices to sum the expected value of the scores for each question. We see that the sum is

$$\begin{aligned}\sum_{n=1}^{30} n\cdot\left(\frac{1}{n^2}+\frac{1}{n+1}+\frac{1022}{n^2(n+1)}\right) &= \sum_{n=1}^{30}\left(\frac{1}{n}+1-\frac{1}{n+1}+1022\left(\frac{1}{n}-\frac{1}{n+1}\right)\right)\\ &= 30+1023\sum_{n=1}^{30}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= 30+1023\left(\frac{1}{1}-\frac{1}{31}\right)\\ &= 30+1023-33\\ &= 1020.\end{aligned}$$

□

14. What is the greatest common factor of 12345678987654321 and 12345654321?

    Proposed by Evan Chen.

    Answer. $\boxed{12321}$.

    Solution. Observe that the numbers in the problem are 111111111^2 and 111111^2 (with nine and six ones). So we seek $\gcd(111111111, 111111)^2$. By the Euclidean Algorithm, this is $\gcd(111000000, 111111)^2 = \gcd(111, 111111)^2 = 12321$. □

15. Let $\phi=\frac{1+\sqrt{5}}{2}$. A *base-$\phi$ number* $(a_n a_{n-1}\dots a_1 a_0)_\phi$, where $0\le a_n, a_{n-1},\dots,a_0\le 1$ are integers, is defined by

$$(a_n a_{n-1}\dots a_1 a_0)_\phi = a_n\cdot\phi^n + a_{n-1}\cdot\phi^{n-1}+\ldots+a_1\cdot\phi^1+a_0.$$

    Compute the number of base-ϕ numbers $(b_j b_{j-1}\dots b_1 b_0)_\phi$ which satisfy $b_j\ne 0$ and

$$(b_j b_{j-1}\dots b_1 b_0)_\phi = \underbrace{(100\dots 100)_\phi}_{\text{Twenty } 100's}.$$

    Proposed by Yang Liu.

    Answer. $\boxed{1048576}$.

Solution. Since $\phi^2 = \phi + 1$, we can do the following transformation: $100_\phi \to 011_\phi$. I claim that in $\underbrace{(100\ldots100)_\phi}_{\text{Twenty }100's}$, for each of the 20 $100's$, we can choose whether to change it to 011, so this gives at least 2^{20} equivalent numbers. I claim that these are the only ones.

Consider the first 2 $100's$ in the string of twenty. If we apply the transformation to the first 100, it becomes 011, but no matter what we do to the second one, neither 1 in the 011 will have 2 zeroes after it; therefore, we can't apply the transformation anymore.

So our final answer is just $2^{20} = 1048576$. $\square$

16. Let $OABC$ be a tetrahedron such that $\angle AOB = \angle BOC = \angle COA = 90^\circ$ and its faces have integral surface areas. If $[OAB] = 20$ and $[OBC] = 14$, find the sum of all possible values of $[OCA][ABC]$. (Here $[\triangle]$ denotes the area of $\triangle$.)

Proposed by Robin Park.

Answer. $\boxed{22200}$.

Solution. The 3D Pythagorean Theorem (also called De Gua's Theorem) states that the sum of the squares of the areas of the faces of a right-angled tetrahedron adjacent to the right angle is equal to the square of the area of the face opposite to it; in other words,

$$[OAB]^2 + [OBC]^2 + [OCA]^2 = [ABC]^2.$$

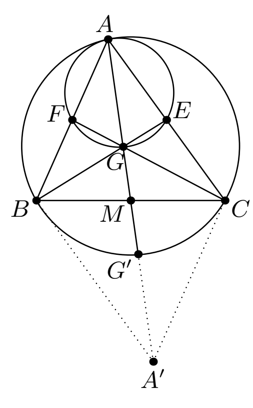
Let $u = [OCA]$ and $v = [ABC]$. Then $(v-u)(v+u) = v^2 - u^2 = 596 = 4 \cdot 149$, and since the parities of $v-u$ and $v+u$ must be the same, $v+u = 2\cdot 149$ and $v - u = 2$. Thus the only integer solution (u, v) is $(150, 148)$, and so our answer is $150 \cdot 148 = 22200$. $\square$

17. Let ABC be a triangle with area 5 and $BC = 10$. Let E and F be the midpoints of sides AC and AB respectively, and let BE and CF intersect at G. Suppose that quadrilateral $AEGF$ can be inscribed in a circle. Determine the value of $AB^2 + AC^2$.

Proposed by Ray Li.

Answer. $\boxed{200}$.

Solution 1. Let M be the midpoint of side BC and let G' be the reflection of G over A. By considering the homothety at A with ratio 2, we see that G' lies on the circumcircle of ABC.



By Power of a Point, we have

$$AM \cdot MG' = BM \cdot MC = 5^2 = 25.$$

But $MG' = MG = \frac{1}{3}AM$ (since G is the centroid), so we derive that $AM = 5\sqrt{3}$.

Now let us reflect A over M to A', yielding parallelogram $ABA'C$. It is a well known fact about parallelograms that we have

$$BC^2 + AA'^2 = 2(AB^2 + AC^2).$$

We know $AA'^2 = 4 \cdot AM^2$, so the answer to the question is $\frac{1}{2}BC^2 + 2AM^2 = 50 + 150 = 200$. $\square$

Solution 2. As before we see that G' lies on the circumcircle of ABC. The barycentric coordinates of G' are thus $G' = (2 : 2 : -1)$. Letting $a = BC$, $b = CA$, $c = AB$ and using the circumcircle formula, we detect

$$-a^2(2)(2) - b^2(-1)(2) - c^2(2)(-1) = 0.$$

Hence, $b^2 + c^2 = 2a^2 = 200$. $\square$

Solution 3. The concyclicity of $AEGF$ implies

$$\angle AEG + \angle AFG = 180^\circ$$

so

$$\angle AEB + \angle AFC = 180^\circ$$

implying that there is a point X on BC such that $\angle AXC = \angle AFC$ and $\angle AXB = \angle AEB$. So $AEXB, AFXC$ are concyclic, implying

$$BC \cdot BX = BF \cdot BA = \frac{1}{2}BA^2$$

$$BC \cdot CX = CE \cdot CA = \frac{1}{2}CA^2$$

so upon adding these together, we recover

$$BC^2 = BC(BX + CX) = \frac{1}{2}(AB^2 + AC^2)$$

Thus our answer is $2BC^2 = 200$. $\square$

Comment. The condition that the area of ABC is 5 is extraneous.

18. We select a real number α uniformly and at random from the interval $(0, 500)$. Define

$$S = \frac{1}{\alpha} \sum_{m=1}^{1000} \sum_{n=m}^{1000} \left\lfloor \frac{m+\alpha}{n} \right\rfloor.$$

Let p denote the probability that $S \ge 1200$. Compute $1000p$.

Proposed by Evan Chen.

Answer. $\boxed{5}$.

Solution. Define $c = \lfloor \alpha \rfloor$. Switch the order of summation:

$$\sum_{a=1}^{1000} \sum_{b=1}^{a} \left\lfloor \frac{b+c}{a} \right\rfloor.$$

One can check that in fact (say, using the Hermite identity) that we have

$$\sum_{b=1}^{a} \left\lfloor \frac{b+c}{a} \right\rfloor = 1 + c$$

for every a.

Hence

$$S = 1000 \cdot \frac{c+1}{c+\epsilon} = 1000 + 1000 \cdot \frac{1-\epsilon}{c+\epsilon},$$

where $\epsilon = \alpha - c$. Hence $S \geq 1200$ is just $5 - 5\epsilon \geq c + \epsilon$ or $\epsilon \leq \frac{5-c}{6}$. Over $c = 0, 1, \ldots, 4$ the intervals for which $S \geq 1200$ sum to $\frac{5}{2}$. Hence $1000p = 1000 \cdot \frac{5}{2} \cdot \frac{1}{500} = 5$. $\square$

19. In triangle ABC, $AB = 3$, $AC = 5$, and $BC = 7$. Let E be the reflection of A over $\overline{BC}$, and let line BE meet the circumcircle of ABC again at D. Let I be the incenter of $\triangle ABD$. Given that $\cos^2 \angle AEI = \frac{m}{n}$, where m and n are relatively prime positive integers, determine $m + n$.

Proposed by Ray Li.

Answer. $\boxed{55}$.

Solution. By construction, C is the midpoint of arc AD not containing B. Hence, it follows that $IC = IA = ID$. Moreover it is easy to see that $\angle IAE = \angle IEA$.

One can use Stewart's theorem to compute $AI^2 = \frac{5^3 \cdot 3 + 2^2 \cdot 5 - 2 \cdot 5 \cdot 7}{7} = \frac{25}{7}$. By Heron's formula, the area of ABC is $\frac{15}{4}\sqrt{3}$, so the height from A to BC has length $\frac{15}{14}\sqrt{3}$. Then

$$\cos^2 \angle IAE = \frac{\left(\frac{15}{14}\sqrt{3}\right)^2}{\frac{25}{7}} = \frac{27}{28}.$$

Thus the answer is $27 + 28 = 55$. $\square$

20. Let $n = 2188 = 3^7 + 1$ and let $A_0^{(0)}, A_1^{(0)}, \ldots, A_{n-1}^{(0)}$ be the vertices of a regular n-gon (in that order) with center O . For $i = 1, 2, \ldots, 7$ and $j = 0, 1, \ldots, n-1$, let $A_j^{(i)}$ denote the centroid of the triangle

$$\triangle A_j^{(i-1)} A_{j+3^{7-i}}^{(i-1)} A_{j+2\cdot 3^{7-i}}^{(i-1)}.$$

Here the subscripts are taken modulo n. If

$$\frac{|OA_{2014}^{(7)}|}{|OA_{2014}^{(0)}|} = \frac{p}{q}$$

for relatively prime positive integers p and q, find $p + q$.

Proposed by Yang Liu.

Answer. $\boxed{2188}$.

Solution. We use vectors/complex numbers. Let $a_j^{(i)}$ be the number at the point $A_j^{(i)}$. Since the centroid is just the average of the vertices,

$$a_j^{(i)} = \frac{1}{3}(a_j^{(i-1)} + a_{j+3^{7-i}}^{(i-1)} + a_{j+2\cdot 3^{7-i}}^{(i-1)}).$$

I claim now that $a_j^{(i)} = \frac{1}{3^i} \sum_{k=0}^{3^i - 1} a_{j+3^{7-i}\cdot k}^{(0)}$. This readily follows from induction and the above statement. So $a_{2014}^{(7)} = \frac{1}{2187} \sum_{k=0}^{3^i-1} a_{j+3^{7-i}\cdot k}^{(0)} = -\frac{1}{2187} a_{2014}^{(0)}$.

Therefore, $\frac{|a_{2014}^{(7)}|}{|a_{2014}^{(0)}|} = \frac{1}{2187} \implies 1 + 2187 = 2188.$ $\square$

21. Consider a sequence $x_1, x_2, \cdots x_{12}$ of real numbers such that $x_1 = 1$ and for $n = 1, 2, \dots, 10$ let

$$x_{n+2} = \frac{(x_{n+1} + 1)(x_{n+1} - 1)}{x_n}.$$

Suppose $x_n > 0$ for $n = 1, 2, \dots, 11$ and $x_{12} = 0$. Then the value of x_2 can be written as $\frac{\sqrt{a}+\sqrt{b}}{c}$ for positive integers a, b, c with $a > b$ and no square dividing a or b. Find $100a + 10b + c$.

Proposed by Michael Kural.

Answer. $\boxed{622}$.

Solution 1. We can rearrange the given condition as

$$x_n x_{n+2} + 1 = x_{n+1}^2.$$

By Ptolemy's theorem, this statement is equivalent to the existence of an isosceles trapezoid $ABCD$ satisfying $AB = CD = 1, AC = BD = x_{n+1}, BC = x_n$, and $AD = x_{n+2}$. So now consider the points $A_0, A_1, \cdots A_{12}$ on a circle with $A_0A_1 = A_1A_2 = \cdots A_{11}A_{12} = 1$, $A_0A_2 = x_2$. Then by Ptolemy's theorem on each trapezoid $A_0A_1A_nA_{n+1}$, it follows inductively that $A_0A_n = x_n$ for each $1 \le n \le 12$. (Note in particular that the condition $x_n \ge 0$ for $1 \le n \le 12$ implies by the form of Ptolemy that each trapezoid is in the "correct" order: that is, there is no n with A_0 between A_n and A_{n+1}.) So $x_{12} = 0$ is equivalent to $A_{12} = A_0$, or $A_0A_1 \cdots A_{11}$ being a regular 12-gon. Thus $A_0A_1A_2$ is a $15° - 15° - 150°$ triangle, and it follows that

$$x_2 = 2\cos 15° = \frac{\sqrt{6}+\sqrt{2}}{2}$$

and the answer is 622. □

Solution 2. Suppose we had $x_2 > 2$, so in particular $x_2 > x_1 + 1$. Note that if $x_{n+1} > x_n + 1$, then

$$x_{n+2} = \frac{(x_{n+1}+1)(x_{n+1}-1)}{x_n} > \frac{(x_{n+1}+1)x_n}{x_n} = x_{n+1} + 1.$$

So inductively, $x_{n+1} > x_n + 1$ for all positive integers n. But this is impossible if $x_{12} = 0$. Thus we must have had $x_2 \le 2$.

So now let $x_2 = 2\cos\alpha$. In particular, $x_2 = \frac{\sin 2\alpha}{\sin\alpha}$ and $x_1 = \frac{\sin\alpha}{\sin\alpha}$. We claim, by induction, that

$$x_n = \frac{\sin n\alpha}{\sin\alpha}$$

We have the trigonometric identity

$$\sin(x)\sin(y) = \sin^2\left(\frac{x+y}{2}\right) - \sin^2\left(\frac{x-y}{2}\right)$$

so in particular

$$\sin n\alpha \sin(n+2)\alpha = \sin^2(n+1)\alpha - \sin^2\alpha$$

$$\Rightarrow \left(\frac{\sin n\alpha}{\sin\alpha}\right)\left(\frac{\sin(n+2)\alpha}{\sin\alpha}\right) = \left(\frac{\sin(n+1)\alpha}{\sin\alpha}\right)^2 - 1$$

and the claim holds by induction.

So now we have $\sin 12\alpha = 0$ and $\sin n\alpha > 0$ for $0 < n < 12$. Thus $12\alpha = 180° \Rightarrow \alpha = 15°$. Thus

$$x_2 = \frac{\sin 30°}{\sin 15°} = \frac{\sqrt{6}+\sqrt{2}}{2}$$

and the answer is 622. □

Comment. If $x_2 = x$, $x_n = U_{n-1}\left(\frac{x}{2}\right)$, where $U_n(x)$ is the nth Chebyshev Polynomial of the second kind.

22. Find the smallest positive integer c for which the following statement holds: Let k and n be positive integers. Suppose there exist pairwise distinct subsets $S_1, S_2, \ldots, S_{2k}$ of $\{1, 2, \ldots, n\}$, such that $S_i \cap S_j \neq \varnothing$ and $S_i \cap S_{j+k} \neq \varnothing$ for all $1 \leq i, j \leq k$. Then $1000k \leq c \cdot 2^n$.

Proposed by Yang Liu.

Answer. $\boxed{334}$.

Solution. Let $S = \{S_1, S_2, \ldots, S_{2k}\}$, and let X be the set of subsets of $\{1, 2, \ldots, n\}$ that are not any of $S_1, S_2, \ldots, S_{2k}$. For any $1 \leq i \leq k$, S_i does not intersect any of $S_1, S_2, \ldots, S_{2k}$, the complement of S_i must lie in X. Therefore, $|X| \geq k$. But $3k = |S| + |X| \leq 2^n$, so a lower bound for c is $\lceil 1000/3 \rceil = 334$. Now we construct the bound. Firstly, note that there exists a constant C such that $\binom{n}{k} \leq C \cdot \frac{2^n}{\sqrt{n}}$ for every n, k (1). Now take an x such that

$$\left(\frac{1}{3} - \frac{C}{\sqrt{n}}\right) \cdot 2^n \leq \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{n-x} \leq \left(\frac{1}{3} + \frac{C}{\sqrt{n}}\right) \cdot 2^n.$$

As n approaches infinity, we can get arbitrarily close to our desired constant of $\frac{1}{3}$, so I'll just use the symbol $\approx$ for the rest of the solution. Then $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{x} = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{n-x} \approx \frac{1}{3} \cdot 2^n$, so
$\binom{n}{n-x-1} + \binom{n}{n-x-2} + \ldots + \binom{n}{x+1} \approx \frac{1}{3} \cdot 2^n$, too. Therefore, just let $S_1, S_2, \ldots, S_k$ be the sets of size $n, n-1, \ldots, n-x$ and $S_{k+1}, S_{k+2}, \ldots, S_{2k}$ be the sets of size $n-x-1, n-x-2, \ldots, x+1$. Each of $S_1, S_2, \ldots, S_k$ are size at least $n-x$ and each of the sets $S_{k+1}, S_{k+2}, \ldots, S_{2k}$ are of size $x+1$. Since $x + 1 + n - x = n + 1 > n$, each pair intersects. So $k \approx \frac{1}{3} \cdot 2^n$, giving the construction. $\square$

23. For a prime q, let $\Phi_q(x) = x^{q-1} + x^{q-2} + \cdots + x + 1$. Find the sum of all primes p such that $3 \leq p \leq 100$ and there exists an odd prime q and a positive integer N satisfying

$$\binom{N}{\Phi_q(p)} \equiv \binom{2\Phi_q(p)}{N} \not\equiv 0 \pmod{p}.$$

Proposed by Sammy Luo.

Answer. $\boxed{420}$.

Solution. Let $N = a_{q-1}p^{q-1} + a_{q-2}p^{q-2} + \cdots + a_1 p + a_0$ be the base-p representation of N. By Lucas' Theorem,

$$\binom{N}{\Phi_q(p)} \equiv \binom{a_{q-1}p^{q-1} + a_{q-2}p^{q-2} + \cdots + a_1 p + a_0}{p^{q-1} + p^{q-2} + \cdots + p + 1} \equiv \binom{a_{q-1}}{1}\binom{a_{q-2}}{1} \cdots \binom{a_1}{1}\binom{a_0}{1}$$

and

$$\binom{2\Phi_q(p)}{N} \equiv \binom{2p^{q-1} + 2p^{q-2} \cdots + 2p + 1}{a_{q-1}p^{q-1} + a_{q-2}p^{q-2} + \cdots + a_1 p + a_0} \equiv \binom{2}{a_{q-1}}\binom{2}{a_{q-2}} \cdots \binom{2}{a_1}\binom{2}{a_0}.$$

Since we do not want either of these to be equivalent to 0 modulo p, each of a_i must equal 1 or 2. Suppose that there are k 2's in the a_i. Then

$$\binom{N}{\Phi_q(p)} \equiv \binom{2\Phi_q(p)}{N} \implies 2^k \equiv 2^{q-k} \pmod{p}.$$

Thus, $\mathrm{ord}_p(2) | p - 2k$. Since $k \leq q$ is arbitrary, a prime p works if and only if $\mathrm{ord}_p(2)$ is odd.

We need $p|2^r - 1$ for some odd r, so 2 must be a quadratic residue mod p. That is, $p \equiv \pm 1 \pmod 8$. If $p \equiv -1 \pmod 8$, setting $r = \frac{p-1}{2}$ works due to Fermat's Little Theorem. Thus, the primes 7, 23, 31, 47, 71, and 79 work. The remaining candidates are primes that are 1 (mod 8), which are 17, 41, 73, 89, and 97. We may check these by seeing if the largest odd factor of $p-1$ works for r. We have $2^9 - 1 = 7 \cdot 73$ and $2^{11} - 1 = 23 \cdot 89$, so 73 and 89 both work, but none of the others do.

Our answer is $7 + 23 + 31 + 47 + 71 + 73 + 79 + 89 = 420$. $\square$

24. Let $\mathcal{A} = A_0A_1A_2A_3 \cdots A_{2013}A_{2014}$ be a *regular 2014-simplex*, meaning the 2015 vertices of $\mathcal{A}$ lie in 2014-dimensional Euclidean space and there exists a constant $c > 0$ such that $A_iA_j = c$ for any $0 \le i < j \le 2014$. Let $O = (0, 0, 0, \dots, 0)$, $A_0 = (1, 0, 0, \dots, 0)$, and suppose A_iO has length 1 for $i = 0, 1, \dots, 2014$. Set $P = (20, 14, 20, 14, \dots, 20, 14)$. Find the remainder when

$$PA_0^2 + PA_1^2 + \cdots + PA_{2014}^2$$

is divided by 10^6.

Proposed by Robin Park.

Answer. $\boxed{348595}$.

Solution 1. Denote by $\overrightarrow{A_k}$ the vector $\overrightarrow{OA_k}$ and by $\overrightarrow{P}$ the vector $\overrightarrow{OP}$, where O is the origin. Note that

$$\begin{aligned}
PA_0^2 + PA_1^2 + \cdots + PA_{2014}^2 &= \sum_{k=0}^{2014} \left((\overrightarrow{A_k} - \overrightarrow{P}) \cdot (\overrightarrow{A_k} - \overrightarrow{P}) \right) \\
&= \sum_{k=0}^{2014} \left(\overrightarrow{A_k} \cdot \overrightarrow{A_k} - 2\overrightarrow{A_k} \cdot \overrightarrow{P} + \overrightarrow{P} \cdot \overrightarrow{P} \right) \\
&= \sum_{k=0}^{2014} |\overrightarrow{A_k}|^2 + \sum_{k=0}^{2014} |\overrightarrow{P}|^2 - 2\overrightarrow{P} \cdot \sum_{k=0}^{2014} \overrightarrow{A_k} \\
&= 2015 + 2015(20^2 + 14^2 + 20^2 + 14^2 + \cdots + 20^2 + 14^2) - 0 \\
&= 2015(1 + 1007(20^2 + 14^2)) = 1209348595
\end{aligned}$$

because a regular simplex is symmetric. Hence our answer is 348595. $\square$

Solution 2. Define $f(P) = (P, \sigma_1) - (P, \sigma_2)$, where (P, σ) denotes the power of P with respect to 2013-sphere σ. Let σ_1 be the 2013-sphere centered at P with radius r and let σ_2 be the unit 2013-sphere centered at the origin O. We use the following lemma:

Lemma 1. The function $f(P)$ is linear.

Proof. Let σ_1 be centered at $(a_1, a_2, \dots, a_{2014})$ with radius r_1, and let σ_2 be centered at $(b_1, b_2, \dots, b_{2014})$ with radius r_2. If $P = (x_1, x_2, \dots, x_{2014})$, then

$$\begin{aligned}
f(P) &= (P, \sigma_1) - (P, \sigma_2) \\
&= (x_1 - a_1)^2 + \cdots + (x_{2014} - a_{2014})^2 - (x_1 - b_1)^2 - \cdots - (x_{2014} - b_{2014})^2 - r_1^2 + r_2^2 \\
&= \sum_{i=1}^{2014} (-2a_ix_i + 2b_ix_i + a_i^2 - b_i^2) - r_1^2 + r_2^2.
\end{aligned}$$

The quadratic terms cancel, and so we are left with the linear terms of x_i, implying that f is linear. $\square$

Note that

$$\begin{aligned}\sum_{i=0}^{2014} PA_i^2 = \sum_{i=0}^{2014}(f(A_i)+r^2) &= 2014r^2 + f\left(\sum_{i=0}^{2014} A_i\right)\\ &= 2014r^2 + 2015f\left(\frac{\sum_{i=0}^{2014} A_i}{2015}\right)\\ &= 2014r^2 + 2015f(O)\\ &= 2014r^2 + 2015((1007(20^2+14^2)-r^2)-(0^2-1^2))\\ &= 1209348595.\end{aligned}$$

Hence the answer is 348595. □

25. Kevin has a set S of 2014 points scattered on an infinitely large planar gameboard. Because he is bored, he asks Ashley to evaluate

$$x = 4f_4 + 6f_6 + 8f_8 + 10f_{10} + \cdots$$

while he evaluates

$$y = 3f_3 + 5f_5 + 7f_7 + 9f_9 + \cdots,$$

where f_k denotes the number of convex k-gons whose vertices lie in S but none of whose interior points lie in S. However, since Kevin wishes to one-up everything that Ashley does, he secretly positions the points so that $y - x$ is as large as possible, but in order to avoid suspicion, he makes sure no three points lie on a single line. Find $|y - x|$.

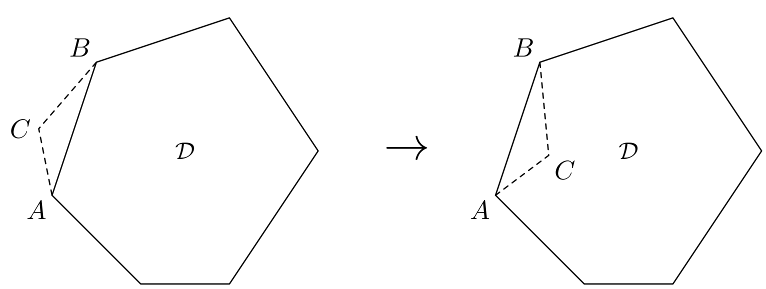
Proposed by Robin Park.

Answer. $\boxed{4054179}$.

Solution. Let A and B be arbitrary points in S. Define $f_{k,AB}$ as the number of empty convex k-gons $\mathcal{C}$ lying on the "right" of line AB, where AB is an edge of $\mathcal{C}$. Then

$$3f_3 - 4f_4 + 5f_5 - 6f_6 + \cdots = \sum_{A,B\in S}(f_{3,AB} - f_{4,AB} + f_{5,AB} - f_{6,AB} + \cdots)$$

because any given convex k-gon will be counted k times, once for each one of its edges.



We claim that the value of $f_{3,AB} - f_{4,AB} + f_{5,AB} - \cdots$ only depends on the number of points that lie to the right of AB. We prove this by moving a point C through AB and noticing that the desired value stays constant. Let $\mathcal{D}$ be a convex k-gon containing AB as an edge. When we move C through segment AB, the number of k-gons decreases by 1 ($\mathcal{D}$ itself), but the number of $(k+1)$-gons also decreases by 1 ($\mathcal{D}$ adjoined to C); thus, $f_{3,AB} - f_{4,AB} + f_{5,AB} - \cdots$ stays constant. In addition, if C moves around on the left without crossing AB, a similar argument shows that the value stays the same. Hence we may assume that A, B, and all p points lying to the right of line AB form a convex k-gon, implying that

$$f_{3,AB} - f_{4,AB} + f_{5,AB} - f_{6,AB} + \cdots = \binom{p}{1} - \binom{p}{2} + \binom{p}{3} - \cdots$$

which is 0 only when $p = 0$ and 1 when $p > 0$. Thus,

$$3f_3 - 4f_4 + 5f_5 - 6f_6 + \cdots = 2\binom{n}{2} - C_S,$$

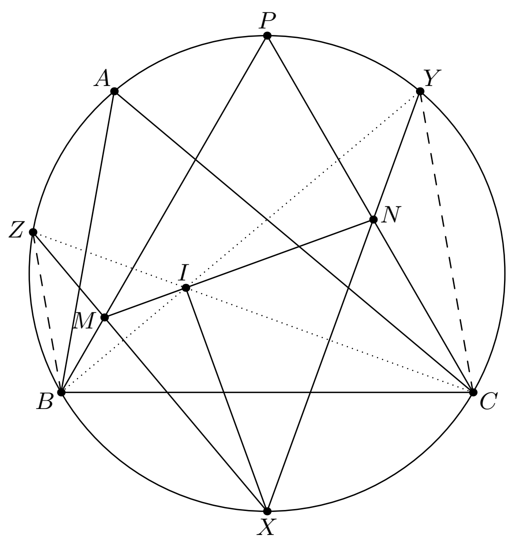
where n is the number of points in S and C_S is the number of points on the convex hull of S. In this problem, $n = 2014$ and $\min C_S = 3$, so our answer is $2\binom{2014}{2} - 3 = 4054179$. □

26. Let ABC be a triangle with $AB = 26$, $AC = 28$, $BC = 30$. Let X, Y, Z be the midpoints of arcs BC, CA, AB (not containing the opposite vertices) respectively on the circumcircle of ABC. Let P be the midpoint of arc BC containing point A. Suppose lines BP and XZ meet at M , while lines CP and XY meet at N. Find the square of the distance from X to MN.

Proposed by Michael Kural.

Answer. $\boxed{325}$.

Solution. Let I be the incenter. By Pascal's Theorem on hexagon $PBYXZC$, I lies on MN.



Consider $\triangle BIZ$, which is isosceles. We have $\angle MIB = \angle MBI = \angle PBY$, and one can check this is equal to $\angle ICB$. Hence MI is tangent to the circumcircle of triangle BIC, as is NI. So MN is the tangent. Since X is the desired circumcenter, we have that I is the foot of the altitude from X to MN.

Now we can compute $IX = BX = BC \cdot \frac{\sin \frac{1}{2}\angle A}{\sin A} = \frac{15}{\cos \frac{1}{2}\angle A}$. Standard computations give that $\cos \frac{1}{2}\angle A = \frac{3}{\sqrt{13}}$, hence $BX^2 = 325$. □

Comment. The use of Pascal is actually not necessary for the solution. Noting that MI, IN are both tangent to the circumcircle of BIC shows that M, I, N are collinear, and that MN is the tangent.

27. Let $p = 2^{16} + 1$ be a prime, and let S be the set of positive integers not divisible by p. Let $f : S \to \{0, 1, 2, \ldots, p-1\}$ be a function satisfying

$$f(x)f(y) \equiv f(xy) + f(xy^{p-2}) \pmod{p} \quad \text{and} \quad f(x+p) = f(x)$$

for all $x, y \in S$. Let N be the product of all possible nonzero values of $f(81)$. Find the remainder when when N is divided by p.

Proposed by Yang Liu and Ryan Alweiss.

Answer. $\boxed{16384}$.

Solution. Let $y^{-1} = 1/y$ denote the inverse of $y \pmod p$. From now on, we'll work in $\mathbb{Z}/p\mathbb{Z}$, or $\mathbb{F}_p$. Then our functional equation is equivalent to $f(x)f(y) = f(xy) + f(x/y)$, since $y^{p-2} = y^{-1}$ by Fermat's Little Theorem. Let $P(x, y)$ be the assertion $f(x)f(y) = f(xy) + f(x/y)$.

I claim that all functions satisfying this equation are of the form $f(x) = x^s + x^{-s}$ for integers s.

Firstly, the assertion $P(1, 1)$ gives that $f(1)^2 = 2f(1)$, so either $f(1) = 0$ or $f(1) = 2$. If $f(1) = 0$, then the assertion $P(x, 1)$ gives $0 = 2f(x) \implies f(x) = 0$ for all x. Since we only care about the product of the nonzero values, we can ignore this case. So $f(1) = 2$.

Lemma: 3 is a primitive root $\pmod p$.

It suffices to show that $3^{2^{15}} \equiv -1 \pmod p$. This follows from

$$3^{2^{15}} \equiv \left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) = -1$$

, by the Quadratic Reciprocity Law.

Now, let k be a positive integer, and look at the assertion $P(3^k, 3)$. This gives $P(3^k)P(3) = P(3^{k+1}) + P(3^{k-1})$ (*), for any integer $k \geq 1$. Let $f(3) = g$, and find an ω such that $f(3) = \omega + \omega^{-1}$. We can solve, using the quadratic formula that $\omega = \frac{g+\sqrt{g^2-4}}{2}$.

If $g^2 - 4$ is a perfect square $\pmod p$, then ω is in $\mathbb{F}_p$. We have another case, though, where $g^2 - 4$ is not a perfect square $\pmod p$. Let $c = g^2 - 4$. In this case, consider all numbers of the form $a + b\sqrt{c}$, where $0 \leq a, b \leq p - 1$. Since c is not a square, all these numbers are distinct. Also, $(a + b\sqrt{c})(d + e\sqrt{c}) = ad + bec + (ae + bd)\sqrt{c}$, so these numbers are closed under multiplication. Since they are obviously also closed under addition, these numbers form a field of order p^2, so we'll just call this set of numbers $\mathbb{F}_{p^2}$. We'll work in this field from now on.

Using $f(1) = 2, f(3) = \omega+\omega^{-1}$, we can use induction and (*) to show that $f(3^k) = \omega^k+\omega^{-k}$. Choosing $k = p - 1$, we get that $2 = f(1) = f(3^{p-1}) = \omega^{p-1} + \omega^{-(p-1)}$. Solving gives that $\omega^{p-1} = 1$, so ω satisfies the polynomial $x^{p-1} - 1$. Since the polynomial can have at most $p - 1$ roots in $\mathbb{F}_{p^2}$, and the integers $1, 2, \ldots, p - 1$ are roots, ω must lie in $\mathbb{F}_p$, contradicting our assumption that $g^2 - 4$ is not a perfect square.

So $f(3) = \omega + \omega^{-1}$ for some $\omega \in \mathbb{F}_p$. Since 3 is a primitive, root, we can find a k such that $3^k = \omega$, so $f(3) = 3^k + 3^{-k}$ for some k. Therefore, $f(81) = 3^{4k} + 3^{-4k}$. These values are distinct for $k = 0, 1, \ldots, \frac{p-1}{8}$, excluding the value $\frac{p-1}{16}$, so it suffices to compute

$$\prod_{0\leq i\leq \frac{p-1}{8}, i\neq \frac{p-1}{16}} (3^{4i} + 3^{-4i}) = -2 \cdot \prod_{0\leq i< \frac{p-1}{8}, i\neq \frac{p-1}{16}} \frac{(3^{8i} + 1)}{3^{4i}}.$$

To evaluate this sum, consider the polynomial

$$\prod_{0\leq i< \frac{p-1}{8}, i\neq \frac{p-1}{16}} \frac{(3^{8i} - x)}{3^{4i}}.$$

For $0 \leq i < \frac{p-1}{8}$, 3^{8i} are roots of this polynomial except when $3^{8i} = -1$, this numerator simplifies to $\frac{1-x^{\frac{p-1}{8}}}{x+1}$, since it satisfies the same properties, and has the same number of roots. The denominator simplifies to $3^{\frac{(p-1)(p-5)}{8}} \equiv -1 \pmod p$. (Since $2^{15}||\frac{(p-1)(p-5)}{8}$).

So to find our desired product, we must find the value of $\frac{1-x^{\frac{p-1}{8}}}{x+1}$ at $x = -1$. Applying L'Hopital's rule, this easily evaluates to $-\frac{p-1}{8}$. Therefore, our final product is

$$-2 \cdot \prod_{0\leq i< \frac{p-1}{8}, i\neq \frac{p-1}{16}} \frac{(3^{8i} + 1)}{3^{4i}} = -2 \cdot \frac{p-1}{8} \cdot -1 = 16384. \qquad \square$$

28. Let S be the set of all pairs (a,b) of real numbers satisfying $1+a+a^2+a^3 = b^2(1+3a)$ and $1+2a+3a^2 = b^2 - \frac{5}{b}$. Find $A+B+C$, where

$$A = \prod_{(a,b)\in S} a, \quad B = \prod_{(a,b)\in S} b, \quad \text{and} \quad C = \sum_{(a,b)\in S} ab.$$

Proposed by Evan Chen.

Answer. $\boxed{2}$.

Solution. First remark that $(a,b) = (1,-1)$ is a solution. Let $z = a - bi$. Adding the first equation to $-bi$ times the second equation yields $z^3 + z^2 + z = -1 + 5i$. Factoring out $z - (1+i)$ since $(1,-1)$ was a solution, we obtain $z^2 + (2+i)z + (2+3i) = 0$. Use Vieta now; if the remaining solutions are (a_1, a_2) and (b_1, b_2) then we know that

$$\begin{aligned} a_1 + a_2 &= -2 \\ b_1 + b_2 &= 1 \\ a_1a_2 - b_2b_1 &= 2 \\ a_1b_2 + a_2b_1 &= -3 \end{aligned}$$

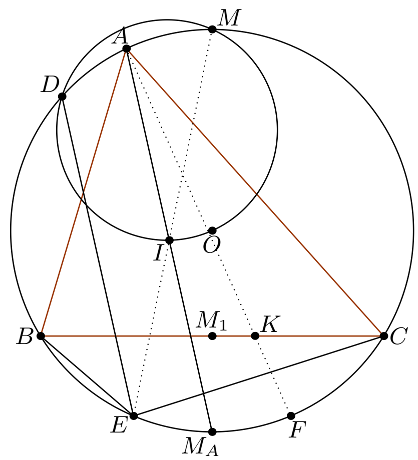
From here it is easy to derive $a_1b_1 + a_2b_2 = (-2)(1) - (-3) = 1$. Thus $C = 1 + 1(-1) = 0$. Furthermore, $A + B = a_1a_2(1) + b_1b_2(-1) = 2$. $\square$

29. Let ABC be a triangle with circumcenter O, incenter I, and circumcircle Γ. It is known that $AB = 7$, $BC = 8$, $CA = 9$. Let M denote the midpoint of major arc $\widehat{BAC}$ of Γ, and let D denote the intersection of Γ with the circumcircle of $\triangle IMO$ (other than M). Let E denote the reflection of D over line IO. Find the integer closest to $1000 \cdot \frac{BE}{CE}$.

Proposed by Evan Chen.

Answer. $\boxed{467}$.

Solution. Let M_A be the midpoint of minor arc BC.



Let $a = BC$, $b = CA$, $c = AB$ and observe that $2a = b + c$. By Ptolemy's Theorem, we obtain

$$BC \cdot AM_A = (AB + AC) \cdot IM_A$$

where we have used the fact that $IM_A = IB = IC$. It follows that $IM_A = \frac{1}{2}AM_A$, so I is the midpoint of $\overline{AM_A}$. (This was also **HMMT February Team Round 2013, Problem 6**).

Now let the tangents to the circumcircle of ABC at M meet line AI at X (not shown). Evidently X lies on the circumcircle of $\triangle IMO$, since $\angle OMX = \angle OIX = 90^\circ$. So XD is also a tangent to the circumcircle of ABC.

Next, notice that $\overline{DE} \parallel \overline{AM_A}$, as both lines are perpendicular to line OI. So it follows that by **JMO 2011 Problem 5** that line EM bisects line AM_A, meaning that E, I, M are collinear. That means E is the tangency point of the A-mixtilinear incircle with the circumcircle of ABC, by an **Iran 2002** problem[2].

Now let the A-excircle be tangent to $\overline{BC}$ at K. Then from **EGMO 2013, Problem 5**,[3] we find that $\angle CAK = \angle BAE$. So if we extend ray AK to meet the circumcircle at F, we get that $BE = CF$, $BF = CE$.

Let $s = \frac{1}{2}(a+b+c)$. We have $\frac{BK}{CK} = \frac{s-c}{s-b}$, from which it follows that $\frac{CF}{BF} = \frac{c(s-b)}{b(s-c)}$. Hence $1000\frac{BE}{CE} = 1000 \cdot \frac{CF}{BF}$, so the requested ratio is.

$$1000 \cdot \frac{c(s-b)}{b(s-c)} = 1000 \cdot \frac{7 \cdot 3}{9 \cdot 5} = 466 + \frac{2}{3}$$

and the answer is 467. $\square$

30. Let $p = 2^{16} + 1$ be an odd prime. Define $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. Compute the remainder when

$$(p-1)! \sum_{n=1}^{p-1} H_n \cdot 4^n \cdot \binom{2p-2n}{p-n}$$

is divided by p.

Proposed by Yang Liu.

Answer. $\boxed{32761}$.

Solution. For the whole solution, $=$ means that the 2 quantities are equal, and $\equiv$ means that I have used the $\pmod p$ operation.

The first step is to notice that $1 + pH_n \equiv \binom{p+n}{n} \pmod{p^2}$. This is true because $\binom{p+n}{n} = \frac{p+n}{n} \cdot \frac{p+n-1}{n-1} \cdot \ldots \cdot \frac{p+1}{1} = \left(1 + \frac{p}{n}\right)\left(1 + \frac{p}{n-1}\right) \ldots \left(1 + \frac{p}{1}\right) \equiv 1 + pH_n \pmod{p^2}$.

Let

$$S = \sum_{n=1}^{p-1} H_n \cdot 4^n \cdot \binom{2p-2n}{p-n}.$$

Then

$$pS = \sum_{n=1}^{p-1} pH_n \cdot 4^n \cdot \binom{2p-2n}{p-n} = \sum_{n=1}^{p-1} (1+pH_n) \cdot 4^n \cdot \binom{2p-2n}{p-n} - \sum_{n=1}^{p-1} 4^n \cdot \binom{2p-2n}{p-n}$$

$$\equiv \sum_{n=1}^{p-1} \binom{p+n}{n} \cdot 4^n \cdot \binom{2p-2n}{p-n} - \sum_{n=1}^{p-1} 4^n \cdot \binom{2p-2n}{p-n} \pmod{p^2}$$

$$= \sum_{n=0}^{p} \binom{p+n}{n} \cdot 4^n \cdot \binom{2p-2n}{p-n} - \sum_{n=0}^{p} 4^n \cdot \binom{2p-2n}{p-n} - \binom{2p}{p} - 4^p \binom{2p}{p} + \binom{2p}{p} + 4^p \pmod{p^2}$$

[2] See `http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=5131`.
[3] European Girl's Math Olympiad

$$\equiv \sum_{n=0}^{p} \binom{p+n}{n} \cdot 4^n \cdot \binom{2p-2n}{p-n} - \sum_{n=0}^{p} 4^n \cdot \binom{2p-2n}{p-n} - 4^p \pmod{p^2}$$

since $\binom{2p}{p} \equiv 2 \pmod{p}$ by Wolstenholme's Theorem.

Now note that $\binom{p+n}{n} = (-1)^n \cdot \binom{-p-1}{n}$, and $4^n\binom{2p-2n}{p-n} = -4^p \cdot (-1)^n \cdot (-4)^{-(p-n)}\binom{2p-2n}{p-n} = -4^p \cdot (-1)^n \cdot \binom{-1/2}{p-n}$. Therefore, our above sum is equal to

$$-4^p \sum_{n=0}^{p} \binom{-p-1}{n} \cdot \binom{-1/2}{p-n} + 4^p \sum_{n=0}^{p} (-1)^n \cdot \binom{-1/2}{p-n} - 4^p \pmod{p^2}$$

$$\equiv -4^p \binom{\frac{-2p-3}{2}}{p} + 4^p \binom{-3/2}{p} - 4^p$$

by Vandermonde's Convolution Identity.

Let's first focus on $-4^p\binom{\frac{-2p-3}{2}}{p}$. This quantity equals

$$-4^p \cdot \frac{4p+1}{2p+1} \cdot \binom{\frac{-2p-1}{2}}{p} = 2^p \cdot \frac{4p+1}{2p+1} \cdot \frac{\prod_{0 \le i \le p-1} 2p+2i+1}{p!}$$

$$= 3 \cdot 2^p \cdot \frac{4p+1}{2p+1} \cdot \frac{\prod_{0 \le i \le p-1, i \ne \frac{p-1}{2}} 2p+2i+1}{(p-1)!}$$

$$\equiv 3 \cdot 2^p \cdot \frac{4p+1}{2p+1} \cdot \frac{\prod_{0 \le i \le p-1, i \ne \frac{p-1}{2}} (2i+1) + 2p \cdot \left(\prod_{0 \le i \le p-1, i \ne \frac{p-1}{2}} (2i+1)\right) \cdot \left(\sum_{0 \le i \le p-1, i \ne \frac{p-1}{2}} \frac{1}{2i+1}\right)}{(p-1)!}$$

$$\equiv 3 \cdot 2^p \cdot \frac{4p+1}{2p+1} \cdot \frac{\prod_{0 \le i \le p-1, i \ne \frac{p-1}{2}} (2i+1)}{(p-1)!} \pmod{p^2}$$

since

$$\sum_{0 \le i \le p-1, i \ne \frac{p-1}{2}} \frac{1}{2i+1} = \sum_{0 \le i < \frac{p-1}{2}} \frac{2p}{(2i+1)(2p-2i-1)} \equiv 0 \pmod{p}.$$

Continuing from where we left off above,

$$\equiv 3 \cdot 2^p \cdot \frac{4p+1}{2p+1} \cdot \frac{\prod_{0 \le i \le p-1, i \ne \frac{p-1}{2}} (2i+1)}{(p-1)!}$$

$$= 6 \cdot \frac{4p+1}{2p+1} \cdot \binom{2p-1}{p-1} \equiv 6 \cdot \frac{4p^2+4p+1}{2p+1} \equiv 12p+6 \pmod{p^2}.$$

Once again, $\binom{2p-1}{p-1} \equiv 1$ by Wolstenholme's Theorem.

The other quantity is much easier. Indeed,

$$4^p \binom{-3/2}{p} = (2p+1) \cdot 4^p \binom{-1/2}{p} = -(2p+1)\binom{2p}{p} \equiv -4p-2 \pmod{p^2}$$

by Wolstenholme's Theorem.

So $pS \equiv 12p+6-4p-2-4^p = 8p+(4-4^p) \pmod{p^2}$. Therefore, $(p-1)!S \equiv -\frac{8p+(4-4^p)}{p} = \frac{4^p-4}{p} - 8$.

We can compute $\frac{4^p-4}{p}$ in the following way: let $x = 2^{16} \equiv -1 \pmod{p}$, so $\frac{4^p-4}{p} = 4 \cdot \frac{x^{2^{13}}-1}{x+1} \equiv -2^{15} \pmod{x+1}$ by L'Hopital's Rule. Therefore our final answer is $\frac{4^p-4}{p} - 8 \equiv -2^{15} - 8 \equiv 2^{15} - 7 = 32761 \pmod{p}$. $\square$