# The Online Math Open Fall Contest
# Solutions
# September 24-October 1, 2012

Online Math Open

# Contest Information

## Format

The test will start Monday September 24 and end Monday October 1. You will have until 7pm EST on October 1 to submit your answers. The test consists of 30 short answer questions, each of which has a nonnegative integer answer. The problem difficulties range from those of AMC problems to those of Olympiad problems. Problems are ordered in roughly increasing order of difficulty.

## Team Guidelines

Students may compete in teams of up to four people. Participating students must not have graduated from high school. International students may participate. No student can be a part of more than one team. The members of each team do not get individual accounts; they will all share the team account.

Each team will submit its final answers through its team account. Though teams can save drafts for their answers, the current interface does not allow for much flexibility in communication between team members. We recommend using Google Docs and Spreadsheets to discuss problems and compare answers, especially if teammates cannot communicate in person. Teams may spend as much time as they like on the test before the deadline.

## Aids

Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids are not allowed. This is includes (but is not limited to) Geogebra and graphing calculators. **Published print and electronic resources are not permitted.** (This is a change from last year's rules.)

Four-function calculators are permitted on the Online Math Open. That is, calculators which perform only the four basic arithmetic operations (+-*/) may be used. Any other computational aids such as scientific and graphing calculators, computer programs and applications such as Mathematica, and online databases is prohibited. All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.

## Clarifications

Clarifications will be posted as they are answered. For the Fall 2012-2013 Contest, they will be posted at here. If you have a question about a problem, please email OnlineMathOpenTeam@gmail.com with "Clarification" in the subject. We have the right to deny clarification requests that we feel we cannot answer.

## Scoring

Each problem will be worth one point. Ties will be broken based on the highest problem number that a team answered correctly. If there are still ties, those will be broken by the second highest problem solved, and so on.

## Results

After the contest is over, we will release the answers to the problems within the next day. If you have a protest about an answer, you may send an email to OnlineMathOpenTeam@gmail.com (Include "Protest" in the subject). Solutions and results will be released in the following weeks.

**Note.** Some of the solutions were taken from the Art of Problem Solving discussion threads; we have given due credit whenever appropriate.

1. Calvin was asked to evaluate $37 + 31 \times a$ for some number $a$. Unfortunately, his paper was tilted 45 degrees, so he mistook multiplication for addition (and vice versa) and evaluated $37 \times 31 + a$ instead. Fortunately, Calvin still arrived at the correct answer while still following the order of operations. For what value of $a$ could this have happened?

   **Answer:** $\boxed{37}$.

   **Solution.** We have to solve a linear equation $37 + 31a = 37 \times 31 + a$. It is easy to check that $a = 37$ works, and since the equation is linear, it is the only solution. (Alternatively, just solve the equation. :) )

   This problem was proposed by Ray Li

2. Petya gave Vasya a number puzzle. Petya chose a digit $X$ and said, "I am thinking of a three digit number that is divisible by 11. The hundreds digit is $X$ and the tens digit is 3. Find the units digit." Vasya was excited because he knew how to solve this problem, but then realized that the problem Petya gave did not have an answer. What digit $X$ did Petya chose?

   **Answer:** $\boxed{4}$.

   **Solution.** The numbers $\overline{X30}, \overline{X31}, \ldots, \overline{X39}$ must all not be multiples of 11. It follows that $\overline{X29}$ and $\overline{X40}$ must be divisible by 11. From the later it is easy to see that $X$ must be 4.

   This problem was proposed by Ray Li

3. Darwin takes an $11 \times 11$ grid of lattice points and connects every pair of points that are 1 unit apart, creating a $10 \times 10$ grid of unit squares. If he never retraced any segment, what is the total length of all segments that he drew?

   **Clarifications:**

   - The problem asks for the total length of all *unit* segments (with two lattice points in the grid as endpoints) he drew.

   **Answer:** $\boxed{220}$.

   **Solution.** The horizontal segments form 11 rows with 10 unit segments each, making 110 unit horizontal segments. Similarly, there are 110 vertical segments, for 220 unit segments total, and the length is thus 220.

   This problem was proposed by Ray Li.

4. Let $\text{lcm}(a, b)$ denote the least common multiple of $a$ and $b$. Find the sum of all positive integers $x$ such that $x \le 100$ and $\text{lcm}(16, x) = 16x$.

   **Answer:** $\boxed{2500}$.

   **Solution.** The key idea is to note that the condition holds if and only if $x$ is odd. We present one of many arguments for why this is true:

   One can show that for positive integers $a, b$, $\gcd(a, b) \cdot \text{lcm}(a, b) = ab$. From this, it follows that $\gcd(16, x) = 1$, so $x$ must be odd. We thus need to compute the sum of the odd numbers less than 100, which is $50^2 = 2500$.

   In a general form, the fact key fact for the above explanation is that $\gcd(a, b) = 1$ if and only if $\text{lcm}(a, b) = ab$.

   This problem was proposed by Ray Li.

5. Two circles have radius 5 and 26. The smaller circle passes through center of the larger one. What is the difference between the lengths of the longest and shortest chords of the larger circle that are tangent to the smaller circle?

   **Answer:** $\boxed{4}$.

   **Solution.** The longest possible chord is a diameter of the larger circle, so the maximum length is 52. The shortest possible chord is intuitively the chord tangent to the smaller circle parallel to said diameter, which one can compute with pythagorean theorem to be 48. This gives the answer of 4.

   The above was all that was expected of participants.* However, the interested student can consider the following proof that the intuitively shortest chord actually is the shortest chord.

   Let the tangency point of the smaller circle with the chord divide the chord into segments of length $x$ and $y$, and let $d$ be the distance from the tangency point to the center. Clearly, we have $d \le 10$. Now we have, by power of a point $26^2 - d^2 = xy$. But by the above and AM-GM, we have

   $$24^2 = 26^2 - 10^2 \le 26^2 - d^2 = xy \le \left(\frac{x+y}{2}\right)^2$$

   whence $x + y \ge 48$. (The AM-GM inequality states that $2\sqrt{xy} \le x + y$ for positive reals $x, y$.)

   This problem was proposed by Ray Li.

6. An elephant writes a sequence of numbers on a board starting with 1. Each minute, it doubles the sum of all the numbers on the board so far, and without erasing anything, writes the result on the board. It stops after writing a number greater than one billion. How many distinct prime factors does the largest number on the board have?

   **Answer:** $\boxed{2}$.

   **Solution.** After listing a few terms, we notice that the numbers on the board have the following pattern:

   $1, 2, 2 \cdot 3, 2 \cdot 3^2, 2 \cdot 3^3, \ldots$.

   One can show by induction using a geometric series that every term after the first will be twice a power of 3, where the exponent increases by 1 each time. Hence the largest number on the board will have exactly 2 prime factors.

   This problem was proposed by Ray Li.

7. Two distinct points $A$ and $B$ are chosen at random from 15 points equally spaced around a circle centered at $O$ such that each pair of points $A$ and $B$ has the same probability of being chosen. The probability that the perpendicular bisectors of $OA$ and $OB$ intersect strictly inside the circle can be expressed in the form $\frac{m}{n}$, where $m, n$ are relatively prime positive integers. Find $m + n$.

   **Answer:** $\boxed{11}$.

   **Solution.** Notice that the condition does not change if we rotate the circle, so if we label the points $1, 2, \ldots, 15$, we can assume without loss of generality that $A$ is point 1. Now, by identifying equilateral triangles, we notice that the perpendicular bisectors of $OA$ and $OB$ intersect on the circle when $B$ is point 6 or point 11, that is, 120 degrees apart. Now it is easy to see that when B is one of points 6, 7, 8, 9, 10, or 11, then the condition is false, so there are $14-6=8$ possibilities for $B$, so the answer is $\frac{8}{14}=\frac{4}{7} \implies 4+7=11$.

This problem was proposed by Ray Li.

8. In triangle $ABC$ let $D$ be the foot of the altitude from $A$. Suppose that $AD=4$, $BD=3$, $CD=2$, and $AB$ is extended past $B$ to a point $E$ such that $BE=5$. Determine the value of $CE^2$.

   **Clarifications:**

   - Triangle $ABC$ is acute.

   **Answer:** $\boxed{80}$.

   **Solution.** By Pythagorean theorem, $AB=5$ and $AC=\sqrt{20}$ so $5=BA=BC=BE$, but because $B$ is on side $AE$, of triangle $ACE$, triangle $ACE$ is a right angle! (If you have not seen this before, you are encouraged to google Thale's theorem.) Now, we have by the Pythagorean theorem on $\triangle ACE$ that $CE^2=AE^2-AC^2=100-20=80$.

   This problem was proposed by Ray Li.

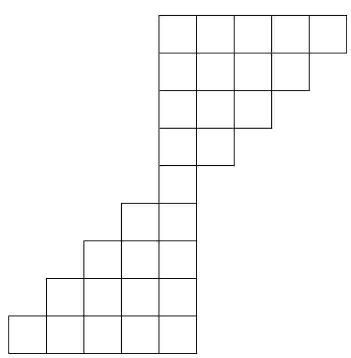
9. Define a sequence of integers by $T_1=2$ and for $n\geq 2$, $T_n=2^{T_{n-1}}$. Find the remainder when $T_1+T_2+\cdots+T_{256}$ is divided by 255.

   **Answer:** $\boxed{20}$.

   **Solution.** Notice that every term from $T_3$ on is a multiple of 8, hence every term from $T_4$ is of the form $2^{8x}=256^x$ for some integer $x$. Hence, $T_4, T_5, \ldots, T_{256}$ all leave a remainder of 1 when divided by 255. Thus, we have $T_1+T_2+\cdots+T_{256}\equiv T_1+T_2+T_3+253\equiv 2+4+16+253\equiv 20 \pmod{255}$, so the answer is 20.

   This problem was proposed by Ray Li.

10. There are 29 unit squares in the diagram below. A frog starts in one of the five (unit) squares on the top row. Each second, it hops either to the square directly below its current square (if that square exists), or to the square down one unit and left one unit of its current square (if that square exists), until it reaches the bottom. Before it reaches the bottom, it must make a hop every second. How many distinct paths (from the top row to the bottom row) can the frog take?



**Answer:** $\boxed{256}$.

**Solution.** Suppose instead of only jumping down, the frog made two paths: One from the center square to the top row, one from the center square to the bottom row. (We are essentially reversing the frog's jumps in the upper half.) Call these paths "small" paths and the paths from top row to bottom row "big" paths. The number of pairs of such small paths is equal to the total number of big paths, because each big path is just two small paths joined together. Notice that for each small path, at any point, the frog has exactly 2 options on which square he can jump to. Thus, each small path has 16 possibilities, giving $16^2 = 256$ total possibilities.

This problem was proposed by Ray Li.

11. Let $ABCD$ be a rectangle. Circles with diameters $AB$ and $CD$ meet at points $P$ and $Q$ inside the rectangle such that $P$ is closer to segment $BC$ than $Q$. Let $M$ and $N$ be the midpoints of segments $AB$ and $CD$. If $\angle MPN = 40^\circ$, find the degree measure of $\angle BPC$.

**Answer:** $\boxed{160}$.

**Solution.** Notice $PQ$ is parallel to $AB$, and by symmetry $PQ$ bisects $\angle MPN$, so $\angle BMP = \angle MPQ = \angle QPN = \angle PNC = \frac{\angle MPN}{2} = 20^\circ$. Because $M$ and $N$ are the circumcenters of the two circles, we have $MB = MP$ and $NP = NC$, so $\angle MBP = \angle MPB = \angle NPC = \angle NCP = 80^\circ$, whence $\angle PBC = \angle PCB = 90 - 80 = 10^\circ$, so $\angle BPC = 180 - \angle PBC - \angle PCB = 160^\circ$.

This problem was proposed by Ray Li.

12. Let $a_1, a_2, \ldots$ be a sequence defined by $a_1 = 1$ and for $n \geq 1$, $a_{n+1} = \sqrt{a_n^2 - 2a_n + 3} + 1$. Find $a_{513}$.

**Answer:** $\boxed{33}$.

**Solution.** One can rearrange the recursion to $(a_{n+1} - 1)^2 = (a_n - 1)^2 + 2$. Thus, the sequence $(a_1 - 1)^2, (a_2 - 1)^2, \ldots$ is an increasing sequence of consecutive even numbers, starting at $(a_1 - 1)^2 = 0$. Thus, $(a_n - 1)^2 = 2n - 2$, so the $(a_{513} - 1)^2 = 1024$, and $a_{513} = 33$.

This problem was proposed by Ray Li.

13. A number is called *6-composite* if it has exactly 6 composite factors. What is the 6th smallest 6-composite number? (A number is *composite* if it has a factor not equal to 1 or itself. In particular, 1 is not composite.)

**Answer:** $\boxed{441}$.

**Solution.** We will casework on the number of prime factors of the number. Notice that if a 6-composite number has $k$ prime factors, then it must have $7 + k$ total factors (primes, composites, and 1)

If the number has 1 prime factor, then the number must have 8 factors, so it must be of the form $p^7$. The first few of these are $2^7 = 128, 3^7, 5^7$.

If the number has 2 prime factors, then the number must have 9 factors, so it must be of the form $p^2q^2$, or $(pq)^2$ (It cannot be $p^8$ or $q^8$ because then the other prime disappears!) The first few terms are $6^2 = 36, 10^2 = 100, 14^2 = 196, 15^2 = 225, 21^2 = 441$.

If a number has 3 prime factors, there must be 10 factors total. However, this means that if the prime powers in the prime factorization are $x, y, z$, then $(x + 1)(y + 1)(z + 1) = 10$, and $x, y, z \geq 1$, and this is impossible, so no solutions here.

If a number has $k \geq 4$ prime factors, then the number must have at least $2^k$ factors, which is always bigger than $7 + k$ for $k \geq 4$, so no solutions here.

Merging our two sets of solutions together, we find that 441 is the sixth smallest element.

This problem was proposed by Ray Li.

14. When Applejack begins to buck trees, she starts off with 100 energy. Every minute, she may either choose to buck $n$ trees and lose 1 energy, where $n$ is her current energy, or rest (i.e. buck 0 trees) and gain 1 energy. What is the maximum number of trees she can buck after 60 minutes have passed?

**Clarifications:**

- The problem asks for the maximum *total* number of trees she can buck in 60 minutes, not the maximum number she can buck on the 61st minute.
- She does not have an energy cap. In particular, her energy may go above 100 if, for instance, she chooses to rest during the first minute.

**Answer:** $\boxed{4293}$.

**Solution.** The key observation is that if she ever rests immediately before she bucks, switching the two operations will strictly increase the number of trees she bucks. It follows that all the resting should be done at the beginning. Now we just need to compute how many times she should rest.

Let's suppose she rests $n$ times. Then she bucks $60 - n$ times, bucking $100 + n, 99 + n, \ldots, (100 + n) - (60 - n) + 1 = 41 + 2n$ trees at the respective times*. Thus, because this is an arithmetic sequence, we can compute the sum to be $\frac{(100+n)+(41+2n)}{2}(60 - n)$ trees. This is a quadratic in $n$, and we can find the maximum to occur at $n = 6.5$, so she should either rest 6 or 7 times, both of which yield an answer of 4293. (In fact, we only have to plug in one of 6 and 7 since we know a priori by the symmetry of a parabola about its vertex that the values will be the same.)

This problem was proposed by Anderson Wang.

15. How many sequences of nonnegative integers $a_1, a_2, \ldots, a_n$ $(n \geq 1)$ are there such that $a_1 \cdot a_n > 0$, $a_1 + a_2 + \cdots + a_n = 10$, and $\displaystyle\prod_{i=1}^{n-1}(a_i + a_{i+1}) > 0$?

**Clarifications:**

- If you find the wording of the problem confusing, you can use the following, equivalent wording: "How many finite sequences of nonnegative integers are there such that (i) the sum of the elements is 10; (ii) the first and last elements are both positive; and (iii) among every pair of adjacent integers in the sequence, at least one is positive."

**Answer:** $\boxed{19683}$.

**Solution.** We will biject the number of sequences to the following:

Consider 10 balls in a row. There are 9 spaces in between the balls. In each space, we do one of three things: i) Stick a "space divider"; ii) Stick a "divider"; iii) Do nothing.

We now let the lengths of consecutive sequences of balls with no dividers in between denote the $a_n$'s, and space dividers denote zeros in between the nonzero numbers. (It's simple to show that this is actually a bijection—prove it!)

As an example, let $x$ denote a ball, $D$ denote a divider and $S$ denote a space-divider. Then the sequence $4, 0, 3, 1, 0, 2$ would correspond to $xxxxSxxxDxSxx$ and the sequence 10 would correspond to $xxxxxxxxxx$.

Clearly each of the 9 spaces has 3 options, giving an answer of $3^9 = 19683$.

**Comment.** Due to potential ambiguity even after the clarification, both $3^9 - 1 = 19682$ and $3^9 = 19683$ were accepted as correct answers.

This problem was proposed by Ray Li.

16. Let $ABC$ be a triangle with $AB = 4024$, $AC = 4024$, and $BC = 2012$. The reflection of line $AC$ over line $AB$ meets the circumcircle of $\triangle ABC$ at a point $D \neq A$. Find the length of segment $CD$.

**Answer:** $\boxed{3521}$.

**Solution.** Let $P$ be the intersection of $AC$ and $BD$. One can angle chase to find $\triangle ADP \sim \triangle ABC$ by AA similarity, so $ADP$ is iscoseles and $AD = AP = 2(DP)$. But $\triangle ADP \sim CBP$ so $2012 = CB = CP = 2(BP)$, so $BP = 1006$. Hence $AP = AB - BP = 3018$, and $DP = \frac{AP}{2} = 1509$, so $CD = DP + CP = 1509 + 2012 = 3521$. (No algebra necessary!)

This problem was proposed by Ray Li.

17. Find the number of integers $a$ with $1 \leq a \leq 2012$ for which there exist nonnegative integers $x, y, z$ satisfying the equation
$$x^2(x^2 + 2z) - y^2(y^2 + 2z) = a.$$

**Clarifications:**

- $x, y, z$ are not necessarily distinct.

**Answer:** $\boxed{1256}$.

**Solution.** First we can note that setting $y = 0$ and $x = 1$ gives $2z + 1$, so all odd numbers are possible. Additionally, setting $y = 0$ and $x = 2$ gives $4(4 + 2z) = 16 + 8z$, so all positive multiples of 8 strictly greater than 8 are valid. This gives 1006 odd numbers plus $2008/8 - 1 = 250$ for a total of 1256 possible numbers.

Now we show that these are all possible values of $a$. Notice that the left side of the equation factors as $(x^2 + y^2 + 2z)(x + y)(x - y)$, so either all the factors are even or all of them are odd. Hence, if the expression is not odd, then it must be a multiple of 8, showing that only odd numbers and multiples of 8 are attainable.

Now it just remains to show 8 is not attainable. Clearly $x \geq y$. and $x - y$ must be at least 2. Then, $x \geq 2$, so $x^2 \geq 4$, and the whole product is at least $(x^2 + y^2 + 2z)(x + y)(x - y) \geq (x^2)(x)(x - y) \geq 4 \cdot 2 \cdot 2 = 16$. Thus, the expression cannot equal 8, so the answer is 1256.

This problem was proposed by Ray Li.

18. There are 32 people at a conference. Initially nobody at the conference knows the name of anyone else. The conference holds several 16-person meetings in succession, in which each person at the meeting learns (or relearns) the name of the other fifteen people. What is the minimum number of meetings needed until every person knows everyone elses name?

**Answer:** $\boxed{6}$.

**Solution.** Let $n = 16$, and suppose for contradiction that 5 meetings suffice. Construct a bipartite graph $S \cup T$ with $S$ the set of 5 $K_n$s and $T$ the vertices of the $K_{2n}$; draw $n$ red edges from fixed $s \in S$ to the $n$ vertices $t \in T$ such that $t \in s$ (abuse of notation here, but the meaning is clear); draw blue edges between any two vertices in $T$ sharing a common $K_n$. We need every two vertices of $T$ to have a blue edge. There are $5n$ red edges, so some $t \in T$ has red-degree less than $5n/2n$, so at most 2, say to $s_1, s_2$. But every $t \neq t$ in $T$ shares a blue edge with $t$, so $N(s_1) \cup N(s_2) = T$. But $\deg s_1 = \deg s_2 = n$, $|T| = 2n$, and $t \in N(s_1) \cap N(s_2)$, so we get a contradiction by PIE. Note that the same logic will not work when we change $K_n$ and $K_{2n}$ to $K_n$ and $K_{mn}$ for some $m > 2$.

For the construction, partition the 32 members into 4 groups of 8, and let $G_i \cup G_j$ be the six meetings for $1 \leq i < j \leq 4$.

**Comment.** As noted by Calvin Deng, the answer is in fact the same for all odd $n \geq 3$. The proof that 5 meetings do not suffice still works, but the construction is significantly harder—we leave it to the interested reader. See the discussion so far on AoPS.

This problem was proposed by Victor Wang.

19. In trapezoid $ABCD$, $AB < CD$, $AB \perp BC$, $AB \parallel CD$, and the diagonals $AC$, $BD$ are perpendicular at point $P$. There is a point $Q$ on ray $CA$ past $A$ such that $QD \perp DC$. If

$$\frac{QP}{AP} + \frac{AP}{QP} = \left(\frac{51}{14}\right)^4 - 2,$$

then $\dfrac{BP}{AP} - \dfrac{AP}{BP}$ can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers $m, n$. Compute $m + n$.

**Answer:** $\boxed{61}$.

**Solution.** We will use the following fact multiple times in our solution: If $ABC$ is a right triangle with $D$ the foot of the altitude from the right angle $A$, then $AD^2 = BD \cdot CD$. (Try to prove it if you haven't seen this before.)

Without loss of generality let $AP = 1$, and $BP = r$. Using the above fact on $\triangle ABC$, we have $(CP)(AP) = BP^2$, so $CP = r^2$. Using it on $\triangle BCD$ yields $CP^2 = (BP)(DP)$ so $DP = r^3$, and finally on $\triangle CDQ$, we have $DP^2 = (CP)(PQ)$, so $PQ = r^4$. Thus, our condition becomes

$$r^4 + \frac{1}{r^4} = \left(\frac{51}{14}\right)^4 - 2.$$

We would like to find $r - \frac{1}{r}$, so we can manipulate the above to obtain the answer:

$$\left(r^2 + \frac{1}{r^2}\right)^2 = r^4 + \frac{1}{r^4} + 2 = \left(\frac{51}{14}\right)^4$$
$$r^2 + \frac{1}{r^2} = \frac{51^2}{14^2}$$
$$\left(r - \frac{1}{r}\right)^2 = r^2 + \frac{1}{r^2} - 2 = \frac{51^2 - 2 \cdot 14^2}{14^2} = \frac{47^2}{14^2}$$
$$r - \frac{1}{r} = \frac{47}{14},$$

whence our final answer is $47 + 14 = 61$.

This problem was proposed by Ray Li.

20. The numbers $1, 2, \ldots, 2012$ are written on a blackboard. Each minute, a student goes up to the board, chooses two numbers $x$ and $y$, erases them, and writes the number $2x + 2y$ on the board. This continues until only one number $N$ remains. Find the remainder when the maximum possible value of $N$ is divided by 1000.

    **Answer:** $\boxed{538}$.

    **Solution.** Let $n = 2012$. $N$ is simply the the maximum value of $S = \sum_{k=1}^{n} 2^{a_k} k$ over all $n$-tuples $(a_1, a_2, \ldots, a_n)$ of nonnegative integers satisfying $\sum_{k=1}^{n} 2^{-a_k} = 1$. (Why?)

    Suppose $(a_1, a_2, \ldots, a_n)$ achieves this maximum value, so $a_1 \le a_2 \le \cdots \le a_n$. Assume for contradiction that $(a_1, a_2, \ldots, a_n) \ne (1, 2, \ldots, n-2, n-1, n-1)$; then there exists an index $k$ and a nonnegative integer $i$ such that $a_k = a_{k+1} = a_{k+2} = i$. (Prove this!) Since $2^{-a_k} + 2^{-a_{k+1}} + 2^{-a_{k+2}} \le 1$, $i \ge 2$, so because $\frac{1}{2} + \frac{1}{8} + \frac{1}{8} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$ and

    $$(2^{i-1} - 2^i)k + (2^{i+1} - 2^i)(k+1) + (2^{i+1} - 2^i)(k+2) > 0,$$

    replacing $(a_k, a_{k+1}, a_{k+2})$ with $(a_k - 1, a_{k+1} + 1, a_{k+2} + 1)$ increases $S$, contradiction.

    Finally, we compute

    $$\begin{aligned} N = 2^{n-1} n + \sum_{k=1}^{n-1} 2^k k &= 2^{n-1} n + 2^n (n-2) + 2 \\ &= 2^{n-1}(3n - 4) + 2 \\ &= 2^{2011}(6032) + 2 \\ &\equiv 2^{2016} + 2 \pmod{1000}. \end{aligned}$$

    Since $2^{2016} \equiv 0 \pmod 8$ and $2^{2016} \equiv 2^{16} = 256^2 \equiv 6^2 = 36 \pmod{125}$ (we use $\phi(125) = 100$), we find $N \equiv 536 + 2 = 538 \pmod{1000}$.

    This problem was proposed by Victor Wang.

21. A game is played with 16 cards laid out in a row. Each card has a black side and a red side, and initially the face-up sides of the cards alternate black and red with the leftmost card black-side-up. A

move consists of taking a consecutive sequence of cards (possibly only containing 1 card) with leftmost card black-side-up and the rest of the cards red-side-up, and flipping all of these cards over. The game ends when a move can no longer be made. What is the maximum possible number of moves that can be made before the game ends?

**Answer:** $\boxed{43690}$.

**Solution.** Suppose that instead of colors, we used binary digits, and we said that a red side had a 0 and a black side had a 1. Notice that if we read the binary digits on the cards from left to right, the flipping sequence is equivalent to subtracting a power of 2 from the number. Because it is always possible to subtract 1 (take the rightmost black card and all the red cards to its right), we can use up to the number of moves equal to the binary number at the beginning, which is $2^{15}+2^{13}+\cdots+2^1 = 2\cdot\frac{4^8-1}{3} = 43960$.

**Comment 1.** There is a perhaps more standard "smoothing" solution to this problem that does not explicitly require the binary observation. Let $g(W)$ to be the "flipping" operation on a sequence of face-up sides $b$ and $r$, and define $f(W)$ to be the answer for an initial "word" $W$ of face-up sides $b$ and $r$ (here it's $(br)^8 = brbr\dots br$). Then $f(W) = 1+\max f(Ug(br^k)V)$, where the maximum is taken over all representations of $W$ in the form $Ubr^kV$ for some nonnegative integer $k$ and words $U, V$ (possibly empty). We leave the rest to the interested reader.

**Comment 2.** It may be interesting to consider the problem in which a move must consist of *at least 2 cards*; see the discussion so far on AoPS.

This problem was proposed by Ray Li.

22. Let $c_1, c_2, \dots, c_{6030}$ be 6030 real numbers. Suppose that for any 6030 real numbers $a_1, a_2, \dots, a_{6030}$, there exist 6030 real numbers $\{b_1, b_2, \dots, b_{6030}\}$ such that

$$a_n = \sum_{k=1}^{n} b_{\gcd(k,n)}$$

and

$$b_n = \sum_{d|n} c_d a_{n/d}$$

for $n = 1, 2, \dots, 6030$. Find $c_{6030}$.

**Answer:** $\boxed{528}$.

**Solution.** Let $\phi$ denote Euler's totient function. We have

$$\begin{aligned} a_n = \sum_{k=1}^{n} b_{\gcd(k,n)} &= \sum_{d|n} \phi(d) b_{n/d} \\ &= \sum_{d|n} \phi(d) \sum_{e|n/d} c_e a_{n/de} = \sum_{d|n} a_{n/d} \sum_{e|d} c_e \phi(d/e) \end{aligned}$$

for all choices of $a_i$, so $\sum_{d|n} c_d \phi(n/d) = [n=1]$ for $n = 1, 2, \dots, 6030$. By strong induction we can show that for $n \ge 1$, $c_n = \prod_{p|n}(1-p)$, where the product runs over all distinct primes $p$ dividing $n$.

Indeed, we clearly have $c_1 = 1$, and assuming the result up to $n-1$ for some $n > 1$, it suffices to show that

$$\sum_{d|n} \phi(n/d) \prod_{p|d}(1-p) = 0.$$

But the LHS is simply

$$\begin{aligned}\sum_{S\subseteq\{p:p|n\}} \prod_{p\in S}(1-p) \sum_{d|n,\{p:p|d\}=S} \phi(n/d) &= \sum_{S\subseteq\{p:p|n\}} \phi\left(\frac{n}{\prod_{p\in S} p^{v_p(n)}}\right) \prod_{p\in S} p^{v_p(n)-1}(1-p) \\ &= \sum_{S\subseteq\{p:p|n\}} (-1)^{|S|}\phi(n) = \phi(n)\prod_{p|n}(1-1) = 0,\end{aligned}$$

where we have used the fact that $\sum_{d|m} \phi(d) = m$ for all positive integers $m$.

Finally, we have $c_{6030} = c_{2\cdot 3^2\cdot 5\cdot 67} = (1-2)(1-3)(1-5)(1-67) = 528$.

**Comment.** A good grasp of multiplicative functions (which the proposer expects many contestants used without proof) makes this problem easier to handle. Indeed, if $\phi^{-1}$ denotes the Dirichlet inverse, then $\phi^{-1}(p^k) = 1-p$ for all primes $p$ and positive integers $k$, so the answer is $\phi^{-1}(6030) = \phi^{-1}(2)\phi^{-1}(3^2)\phi^{-1}(5)\phi^{-1}(67) = (1-2)(1-3)(1-5)(1-67) = 528$. Alternatively, one can find a solution using the Dirichlet series $\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$ and the fact that $1 = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} \sum_{n=1}^{\infty} \frac{\phi^{-1}(n)}{n^s}$.

This problem was proposed by Victor Wang.

23. For reals $x \geq 3$, let $f(x)$ denote the function

$$f(x) = \frac{-x + x\sqrt{4x-3}}{2}.$$

Let $a_1, a_2, \ldots$, be the sequence satisfying $a_1 > 3$, $a_{2013} = 2013$, and for $n = 1, 2, \ldots, 2012$, $a_{n+1} = f(a_n)$. Determine the value of

$$a_1 + \sum_{i=1}^{2012} \frac{a_{i+1}^3}{a_i^2 + a_i a_{i+1} + a_{i+1}^2}.$$

**Answer:** $\boxed{4025}$.

**Solution.** First note that $a = \frac{-x+x\sqrt{4x-3}}{2}$ is a root of the quadratic $a^2 + ax + x^2 - x^3 = 0$, so from $a_{n+1} = f(a_n)$ we obtain $a_i^3 = a_i^2 + a_i a_{i+1} + a_{i+1}^2$.

Next, note that

$$\sum_{i=1}^{2012} \frac{a_{i+1}^3}{a_i^2 + a_i a_{i+1} + a_{i+1}^2} - \sum_{i=1}^{2012} \frac{a_i^3}{a_i^2 + a_i a_{i+1} + a_{i+1}^2} = \sum_{i=1}^{2012} \frac{a_{i+1}^3 - a_i^3}{a_i^2 + a_i a_{i+1} + a_{i+1}^2} = \sum_{i=1}^{2012}(a_{i+1} - a_i) = a_{2013} - a_1,$$

so we find that

$$\begin{aligned} a_1 + \sum_{i=1}^{2012} \frac{a_{i+1}^3}{a_i^2 + a_i a_{i+1} + a_{i+1}^2} &= a_{2013} + \sum_{i=1}^{2012} \frac{a_i^3}{a_i^2 + a_i a_{i+1} + a_{i+1}^2} \\ &= 2013 + \sum_{i=1}^{2012} 1 \\ &= 2013 + 2012 \\ &= 4025. \end{aligned}$$

This problem was proposed by Ray Li.

24. In scalene $\triangle ABC$, $I$ is the incenter, $I_a$ is the $A$-excenter, $D$ is the midpoint of arc $BC$ of the circumcircle of $ABC$ not containing $A$, and $M$ is the midpoint of side $BC$. Extend ray $IM$ past $M$ to point $P$ such that $IM = MP$. Let $Q$ be the intersection of $DP$ and $MI_a$, and $R$ be the point on the line $MI_a$ such that $AR \parallel DP$. Given that $\frac{AI_a}{AI} = 9$, the ratio $\frac{QM}{RI_a}$ can be expressed in the form $\frac{m}{n}$ for two relatively prime positive integers $m, n$. Compute $m + n$.

**Clarifications:**

- "Arc $BC$ of the circumcircle" means "the arc with endpoints $B$ and $C$ not containing $A$".

**Answer:** $\boxed{11}$.

**Solution.** First we will show that $Q$ is the centroid of triangle $BCI_a$. Notice that $BICP$ is a parallelogram, so because $BI \perp BI_a$, $CP \perp BI_a$, and similarly, $BP \perp CI_a$. Hence, $P$ is the orthocenter of $BCI_a$. Furthermore, one can angle chase to show that $\angle IBD = \angle BID = \frac{A+B}{2}$, so $BD = ID$, and similarly, $CD = ID$, so $D$ is the circumcenter of triangle $BIC$. But because $\angle IBI_a = \angle ICI_a = 90$, $BICI_a$ is cyclic, so it follows that $D$ is also the circumcenter of $BCI_a$. Hence, $DP$ is the Euler Line of triangle $BCI_a$, and because $MI_a$ is a median, $Q$ is the centroid.

WLOG, $AI = 1$ and $AI_a = 9$. Now, $DI = DI_a = II_a/2 = 4$. By AA similarity, $\triangle ARI_a \sim \triangle DQI_a$, so $\frac{QI_a}{RI_a} = \frac{DI_a}{AI_a} = \frac{4}{9}$. However, because $Q$ is the centroid of $BCI_a$, $2MQ = QI_a$, so $\frac{MQ}{RI_a} = \frac{2}{9}$, so the answer is $2 + 9 = 11$.

This problem was proposed by Ray Li.

25. Suppose 2012 reals are selected independently and at random from the unit interval $[0, 1]$, and then written in nondecreasing order as $x_1 \le x_2 \le \cdots \le x_{2012}$. If the probability that $x_{i+1} - x_i \le \frac{1}{2011}$ for $i = 1, 2, \ldots, 2011$ can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers $m, n$, find the remainder when $m + n$ is divided by 1000.

**Answer:** $\boxed{601}$.

**Solution 1.** Let $N = 2012$ and $M = \frac{1}{2011}$; we want the probability that $x_{i+1} - x_i \le M$ for $i = 1, 2, \ldots, N - 1$.

We claim that the probability $x_{i+1} - x_i > M$ for all $i \in S$ (where $S$ is a fixed set of indices; call this an $S$-*violating* set of points/reals) is $\max(0, 1 - M|S|)^N$. (*) Indeed, in the nontrivial case $1 - M|S| > 0$, each $S$-violating set corresponds to a unique set of $N$ points from the interval $[0, 1 - M|S|]$ (and vice versa): for each $i \in S$, we simply "remove" the segment $[x_i, x_i + M)$ from the interval $[x_i, x_{i+1})$ (or for the other direction, we "insert" the segment $[x'_i, x'_i + M)$ at the beginning of the interval $[x'_i, x'_{i+1})$).

By (*) and PIE, the desired probability is (using properties of finite differences and $\prod_{j=0}^{N-1}((1-kM)-(1-jM)) = 0$)

$$\begin{aligned}
\sum_{k=0}^{N-1}(-1)^k\binom{N-1}{k}\max(0,1-kM)^N &= \sum_{k=0}^{N-1}(-1)^k\binom{N-1}{k}(1-kM)^N \\
&= \sum_{k=0}^{N-1}(-1)^k\binom{N-1}{k}(1-kM)^{N-1}\sum_{j=0}^{N-1}(1-jM) \\
&= \sum_{k=0}^{N-1}(-1)^k\binom{N-1}{k}(-M)^{N-1}k^{N-1}\left(N - M\frac{N(N-1)}{2}\right) \\
&= M^{N-1}\frac{N}{2}\sum_{k=0}^{N-1}(-1)^{N-1-k}\binom{N-1}{k}k^{N-1} \\
&= M^{N-1}\frac{N}{2}(N-1)!\sum_{k=0}^{N-1}(-1)^{N-1-k}\binom{N-1}{k}\binom{k}{N-1} \\
&= M^{N-1}\frac{N}{2}(N-1)! = \frac{1006 \cdot 2010!}{2011^{2010}}.
\end{aligned}$$

Thus our answer is

$$m + n = 1006 \cdot 2010! + 2011^{2010} \equiv 11^{2010} \equiv \binom{2010}{2}10^2 + \binom{2010}{1}10 + 1 \equiv 601 \pmod{1000}.$$

**Solution 2.** Let the $x_{i+1} - x_i = a_i$. Now we would like to find the volume of solutions such that $x_{2012} = x_1 + a_1 + \cdots + a_{2011} \le 1$ and $(a_1, a_2, \ldots, a_{2011})$ is in a cube side $\frac{1}{2011}$ in 2011-D space. Now, notice that for any point $(a_1, a_2, \ldots, a_{2011})$ the possible solutions for $x_1$ form a segment of size $1 - (a_1 + \cdots + a_{2011})$, so the answer is

$$2012! \cdot \int_0^{\frac{1}{2011}} \cdots \int_0^{\frac{1}{2011}} 1 - (a_1 + \cdots + a_{2011})\, da_1\, da_2\, \ldots da_{2011}$$

We multiply by 2012! because originally the numbers could have been in any order. The 1 integrates to $\frac{1}{2011^{2011}}$ and the other terms integrate to $\frac{1}{2 \cdot 2011^{2012}}$ each. Adding gives that the answer is

$$2012!\left(\frac{1}{2011^{2011}} - \frac{2011}{2 \cdot 2011^{2012}}\right) = \frac{2012!}{2 \cdot 2011^{2011}}$$

And we can finish as in the first solution.

**Solution 3.** Equivalently, we want the probability that $y_1, y_2, \ldots, y_{N-1} \le M$ for a random solution in nonnegative integers to $y_0 + y_1 + \cdots + y_N = 1$, or

$$\frac{[x^1](\int_0^M x^t\, dt)^{N-1}(\int_0^\infty x^t\, dt)^2}{[x^1](\int_0^\infty x^t\, dt)^{N+1}}.$$

But $\int_0^M x^t\ dt = \frac{x^M-1}{\ln x}$ and for positive integers $L$, $(-\ln x)^{-L} = \frac{1}{(L-1)!}\int_0^\infty x^t t^{L-1}\ dt$ (note that this is the gamma function when $x = e^{-1}$, and for $L = 1$ we have $\int_0^\infty x^t\ dt = (-\ln x)^{-1}$). Now by "convolution," we find

$$\begin{aligned}[x^1]\left(\int_0^M x^t\ dt\right)^{N-1}\left(\int_0^\infty x^t\ dt\right)^2 &= [x^1](1-x^M)^{N-1}(-\ln x)^{-(N-1)}(-\ln x)^{-2}\\ &= [x^1](1-x^M)^{N-1}\frac{1}{N!}\int_0^\infty x^t t^N\ dt\\ &= \frac{1}{N!}\sum_{0\le k\le N-1,\frac{1}{M}}(-1)^k\binom{N-1}{k}(1-kM)^N\end{aligned}$$

and

$$[x^1]\left(\int_0^\infty x^t\ dt\right)^{N+1} = [x^1](-\ln x)^{-(N+1)} = [x^1]\frac{1}{N!}\int_0^\infty x^t t^N\ dt = \frac{1}{N!}.$$

We finish in the same way as Solution 1.

**Solution 4.** For any real $r$ and positive number $s$ let $r \pmod s$ denote the value $t \in [0, s)$ such that $\frac{r-t}{s}$ is an integer. For any set of reals $X = \{x_1, x_2, \dots, x_N\}$ with $0 \le x_1 \le \cdots \le x_N \le 1$, let $y_i = x_i \pmod{\frac{1}{N-1}}$ and $\pi_X$ denote the permutation of $\{1, 2, \dots, N\}$ such that $0 \le y_{\pi_X(1)} \le \cdots \le y_{\pi_X(N)} < \frac{1}{N-1}$. (We can arbitrarily break ties in $\pi_X$ if $x_i$ and $x_j$ are congruent modulo $\frac{1}{N-1}$ for some indices $i, j$ with $i \ne j$, as such cases are negligible—note that zero probability doesn't imply impossibility.)

Given a permutation $\pi$, let a *bad pair* be a pair $(i, i+1)$ such that $\pi(i) > \pi(i+1)$. Now fix a permutation $\sigma$ with $K$ bad pairs; then there exist $N - K - 1$ sets $X$ such that $\pi_X = \sigma$. (Why?) If $S(N, K)$ denotes the number of permutations $\sigma$ of $\{1, 2, \dots, N\}$ with $K$ bad pairs, then for a fixed *set* of values $Y(X) = \{y_1, y_2, \dots, y_N\}$, there are

$$A = \sum_{K=0}^{N-1}(N-K-1)S(N,K)$$

sets $X$ such that $0 \le x_1 \le \cdots \le x_N \le 1$ and $x_{i+1} - x_i \le \frac{1}{N-1}$ for all indices $i$. But for any permutation $\sigma$ with $K$ bad pairs, the new permutation $\sigma'$ defined by $\sigma'(i) = N + 1 - \sigma(i)$ has $N - K - 1$ bad pairs. Hence $S(N, K) = S(N, N - K - 1)$, so

$$2A = (N-1)\sum_{K=0}^{N-1}S(N,K) = (N-1)N!$$

by pairing up terms, whence $A = \frac{(N-1)N!}{2}$.

Finally, because there are $(N-1)^N$ sets $X$ such that $0 \le x_1 \le \cdots \le x_N \le 1$ for a fixed set $Y(X) = \{y_1, y_2, \dots, y_N\}$, and $\frac{1}{N-1} \mid 1$ means each set $X$ is equally weighted by symmetry, the desired probability for fixed $Y(X)$ is $L = \frac{A}{(N-1)^N} = \frac{N!}{2(N-1)^{N-1}}$. But this value depends only on $N$ (and in particular, not on $Y(X)$), so the desired probability over all valid sets $X$ is $L$ as well, and we're done.

**Comment.** While the second and fourth solutions are nice, they do not generalize as easily as the first and third. (Why?) The "discrete" analog (see, for instance, this discussion) is perhaps easier to solve, and can be used "asymptotically" to solve the "continuous" version presented here (e.g. by taking a limit).

This problem was proposed by Victor Wang. The second solution was provided by Ray Li. The fourth solution was provided by Sohail Farhangi.

26. Find the smallest positive integer $k$ such that

$$\binom{x+kb}{12} \equiv \binom{x}{12} \pmod{b}$$

for all positive integers $b$ and $x$. (*Note:* For integers $a, b, c$ we say $a \equiv b \pmod{c}$ if and only if $a - b$ is divisible by $c$.)

**Clarifications:**

- $\binom{y}{12} = \frac{y(y-1)\cdots(y-11)}{12!}$ for all integers $y$. In particular, $\binom{y}{12} = 0$ for $y = 1, 2, \ldots, 11$.

**Answer:** $\boxed{27720}$.

**Solution 1.** Note that for fixed $k$ and $b$, $f(x) = \frac{1}{b}\left(\binom{x+kb}{12} - \binom{x}{12}\right)$ is a polynomial in $x$ of degree at most $12 - 1 = 11$. We want to find the smallest $k$ such that $f(\mathbb{Z}) \subseteq \mathbb{Z}$, i.e. $f$ takes on only integer values at integer inputs.

The following observation gives us a clean characterization of such "integer-valued" polynomials.

**Fact.** Suppose $P$ is a polynomial in $x$ of degree at most $d$. Then $P$ is integer-valued if and only if there exist integers $a_0, a_1, \ldots, a_d$ such that

$$P(x) = \sum_{i=0}^{d} a_i \binom{x}{i}.$$

*Proof Outline.* Consider the first $d + 1$ finite differences of $P$ and use Newton interpolation. (See Lemma 1 in the first link for a complete proof.) ■

But by the Vandermonde convolution identity,

$$f(x) = \frac{1}{b}\left(\binom{x+kb}{12} - \binom{x}{12}\right) = \frac{1}{b}\sum_{i=1}^{12}\binom{kb}{i}\binom{x}{12-i},$$

so we just need to find the smallest $k$ such that $\frac{k}{i}\binom{kb-1}{i-1} = \frac{1}{b}\binom{kb}{i} \in \mathbb{Z}$ for all $i \in \{1, 2, \ldots, 12\}$ and positive integers $b$.

We immediately see that $k = \operatorname{lcm}(1, 2, \ldots, 12) = 27720$ works, since then $\frac{k}{i} \in \mathbb{Z}$ for $i = 1, 2, \ldots, 12$ and any $b$.

Proving that $i \mid k$ for $i = 1, 2, \ldots, 12$ is only slightly harder. Suppose otherwise for some fixed $i$; then there exists a positive integer $T$ such that $\binom{kc-1}{i-1} \equiv \binom{k(c+T)-1}{i-1} \pmod{i}$ for all integers $c$ (not necessarily positive). (Why?) Taking $c = 0$ and $b = c + T = T$, we have $\frac{k}{i}\binom{-1}{i-1} \in \mathbb{Z}$ by assumption. But $\binom{-1}{r} = (-1)^r$ for all nonnegative integers $r$, contradiction.

**Solution 2.** We start with a helpful lemma (from math.stackexchange) for the case $b = p$.

**Lemma.** Fix a positive integer $m$ and a prime $p$. Then $\binom{n}{m}$ is periodic modulo $p$ with (minimum) period equal to $p^{1+\lfloor \log_p(m) \rfloor}$.

*Proof (Jyrki Lahtonen).* Let $e = 1+\lfloor \log_p(m) \rfloor$. It is relatively easy to show that if the base $p$ expansions of $n$ and $m$ are $n = \sum_{i\ge 0} n_i p^i$ and $m = \sum_{i=0}^{e-1} m_i p^i$, with $0 \le n_i, m_i < p$ for all $i$, then

$$\binom{n}{k} \equiv \prod_{i\ge 0} \binom{n_i}{m_i} \pmod p$$

by Lucas' theorem. The periodicity then follows from the fact that adding $p^e$ to $n$ leaves the base $p$-digits $n_i$ unchanged for $i < e$, yet for $i \ge e$, $\binom{n_i}{m_i} = 1$ (as $m_i = 0$). But then the minimum period divides $p^e$ and must be a power of $p$, so similar reasoning shows that $p^e$ is in fact the minimum period.■

The lemma immediately shows that $27720 = \operatorname{lcm}(1, 2, \ldots, 12) \mid k$, so it remains to show that $k = \operatorname{lcm}(1, 2, \ldots, 12)$ works.

Using the notation from the proof of the lemma, it now suffices to prove that $p^{1+r} \mid \binom{n+p^{e+r}}{m} - \binom{n}{m} = [t^m](1+t)^n((1+t)^{p^{e+r}} - 1)$ for all $n \ge 0$ and $r \ge 0$. (We leave $n < 0$ to the reader.) As $p^e > m$, it's enough to show that $p^{1+r} \mid \binom{p^{e+r}}{l}$ for $l = 1, 2, \ldots, p^e - 1$. But $\binom{p^{e+r}}{l} = \frac{p^{e+r}}{l}\binom{p^{e+r}-1}{l-1} \equiv 0 \pmod{p^{1+r}}$ since $p^e \nmid l$, so we're done.

**Comment.** In the second solution, we can also finish with the fact that if $f(t)-g(t)$ has all coefficients divisible by $p^s$ for two polynomials $f, g$ with integer coefficients, then $f(t)^p - g(t)^p$ has all coefficients divisible by $p^{s+1}$. (Just write $f(t) = g(t) + ph(t)$ to prove this.) Indeed, it then follows by induction on $r \ge 0$ that $(1+t)^{p^{e+r}} - (1+t^{p^e})^{p^r}$ has all coefficients divisible by $p^{1+r}$.

This problem was proposed by Alex Zhu. The second solution was provided by Lawrence Sun.

27. Let $ABC$ be a triangle with circumcircle $\omega$. Let the bisector of $\angle ABC$ meet segment $AC$ at $D$ and circle $\omega$ at $M \ne B$. The circumcircle of $\triangle BDC$ meets line $AB$ at $E \ne B$, and $CE$ meets $\omega$ at $P \ne C$. The bisector of $\angle PMC$ meets segment $AC$ at $Q \ne C$. Given that $PQ = MC$, determine the degree measure of $\angle ABC$.

**Answer:** $\boxed{80}$.

**Solution.** Let the angles of the triangle be $A, B$, and $C$. We can angle chase to find $\angle ACP, = \angle DCE = \angle DBE = \angle ABM = \angle MBC$. Thus, it follows that arcs $AP, AM$, and $MC$ have the same length, so $AP = AM = MC$. Thus, $AP = PQ$. Now, note that by $\widehat{AP} = \widehat{AM}$, and because $MQ$ is the bisector of $\angle PMC$, $Q$ is the incenter of triangle $PMC$.

Now we can angle chase all the angles in terms of $B$. $\widehat{PAMC} = 3B$, so $\angle PMC = \frac{\widehat{PBC}}{2} = 180 - \frac{3B}{2}$. By $Q$ being the incenter of triangle $MPC$, we have $\angle PQC = 90 + \frac{\angle PMC}{2} = 180 - \frac{3B}{4}$, so $\angle AQP = 180 - \angle PQC = \frac{3B}{4}$. Furthermore, $\angle PAQ = \angle PAC = \angle PMC = 180 - \frac{3B}{2}$. By $AP = PQ$, we have $180 - \frac{3B}{2} = \frac{3B}{4}$. Solving, we find $B = 80$.

This problem was proposed by Ray Li.

28. Find the remainder when

$$\sum_{k=1}^{2^{16}} \binom{2k}{k}(3 \cdot 2^{14}+1)^k (k-1)^{2^{16}-1}$$

is divided by $2^{16}+1$. (*Note:* It is well-known that $2^{16}+1 = 65537$ is prime.)

**Answer:** $\boxed{28673}$.

**Solution.** Let $n = 4$, $p = 2^{2^n}+1 = 65537$, $S$ denote the sum in the problem; observe taht $3 \cdot 2^{14}+1 \equiv 4^{-1} \pmod{p}$ and for $k \neq 1$, $(k-1)^{2^{16}-1} \equiv (k-1)^{-1} \pmod{p}$ by Fermat's little theorem. It's well-known (e.g. from the derivation of the generating function $(1-4x)^{-\frac{1}{2}} = \sum_{r \geq 0} \binom{2r}{r} x^r$) that $\binom{2r}{r} = (-4)^r \binom{-\frac{1}{2}}{r}$ for all $r \geq 0$. Thus

$$S \equiv \sum_{k=2}^{p-1} \binom{2k}{k} 4^{-k}(k-1)^{-1} = \sum_{k=2}^{p-1} (-1)^k \binom{-\frac{1}{2}}{k}(k-1)^{-1} \equiv \sum_{k=2}^{p-1} \frac{(-1)^k}{k-1} \binom{\frac{p-1}{2}}{k} \pmod{p}.$$

By Wilson's theorem, $\binom{p}{i} \equiv \frac{p}{i}(-1)^{i-1} \pmod{p}$ for $i = 1, 2, \ldots, p-1$, so

$$pS \equiv \sum_{k=2}^{p-1} \binom{p}{k-1}\binom{\frac{p-1}{2}}{k} = \binom{\frac{3p-1}{2}}{\frac{p-3}{2}} - \binom{p}{0}\binom{\frac{p-1}{2}}{1} \pmod{p^2}$$

by the Vandermonde convolution identity. Standard computations yield

$$\binom{\frac{3p-1}{2}}{\frac{p-3}{2}} = \prod_{i=1}^{\frac{p-3}{2}} \frac{p + \frac{p+1-2i}{2}}{i} \equiv \prod_{i=1}^{\frac{p-3}{2}} \frac{p+1-2i}{2i} \left(1 + p \sum_{i=1}^{\frac{p-3}{2}} \frac{2}{p+1-2i}\right) \pmod{p^2}.$$

Using Wilson's theorem in the same manner as above, this expression simplifies (using the binomial theorem) to

$$\begin{aligned} \frac{p-1}{2}\left(1 - 2\left(2^{p-1} - \binom{p}{2} - \binom{p}{0}\right)\right) &\equiv \frac{p-1}{2}((2-2^p) - (p-1)) \\ &\equiv 2^{p-1} - 1 - \frac{p^2-2p+1}{2} \equiv 2^{p-1} - \frac{3}{2} + p \pmod{p^2}, \end{aligned}$$

whence $pS \equiv 2^{p-1} - 1 + \frac{p}{2} \pmod{p^2}$. Finally, we have

$$S \equiv \frac{2^{p-1}-1}{p} + \frac{1}{2} \equiv \frac{2^{2^{2^n}}-1}{2^{2^n}+1} + \frac{2^{2^n}+2}{2} \equiv -2^{2^n-n} + 2^{2^n-1} + 1 = 28673 \pmod{p},$$

as desired. (We use the fact that $\frac{x^{2s}-1}{x+1} = x^{2s-1} - x^{2s-2} \pm \cdots + x - 1 \equiv -2s \pmod{x+1}$.)

This problem was proposed by Victor Wang.

29. In the Cartesian plane, let $S_{i,j} = \{(x,y) \mid i \leq x \leq j\}$. For $i = 0, 1, \ldots, 2012$, color $S_{i,i+1}$ pink if $i$ is even and gray if $i$ is odd. For a convex polygon $P$ in the plane, let $d(P)$ denote its pink density, i.e. the fraction of its total area that is pink. Call a polygon $P$ *pinxtreme* if it lies completely in the region

$S_{0,2013}$ and has at least one vertex on each of the lines $x = 0$ and $x = 2013$. Given that the minimum value of $d(P)$ over all non-degenerate convex pinxtreme polygons $P$ in the plane can be expressed in the form $\frac{(1+\sqrt{p})^2}{q^2}$ for positive integers $p, q$, find $p + q$.

**Answer:** $\boxed{2025079}$.

**Solution.** Let $M = 1006$ and $N = 2M + 1 = 2013$.

Suppose $P = A_1A_2 \dots A_mB_1B_2 \dots B_n$, where $A_1, B_n$ lie on $x = 0$ and $A_m, B_1$ lie on $x = N$ with $A_1$ above $B_n$ and $A_m$ above $B_1$. As $d(P)$ is a (positively)-weighted average of (i) $d(A_1A_2 \dots A_m)$, (ii) $d(A_1A_mB_1B_n)$, and (iii) $d(B_1B_2 \dots B_n)$, we just need to consider (i) and (ii) to find the desired minimum ((i) and (iii) are analogous)—in each case, the relevant polygon is convex and pinxtreme.

We first introduce a helpful lemma.

**Lemma.** If $X$ is a (not necessarily pinxtreme) non-degenerate convex polygon contained in $S_{0,N}$ with each of its vertices on a line of the form $x = i$ (where $0 \le i \le N$), then $d(X) = \frac{1}{2}$.

*Proof.* Induction on the number of vertices.■

Immediately from the lemma, we have $d(A_1A_mB_1B_n) = \frac{1}{2}$ in case (ii), so it remains to consider case (i). By a simple induction, one can show that $d(A_1A_2 \dots A_m)$ is a (positively)-weighted average of the $d(A_1A_iA_m)$ for $i = 2, 3, \dots, m-1$. (The interested reader may work the details out.)

The rest is standard computation. If $A_i$ is in a pink region, then $d(A_1A_iA_m) \ge \frac{1}{2}$, so we can assume $A_i$ is in the gray region $S_{2j-1,2j}$ for some $j \in [1, M]$. If $A_j$ lies on the line $x = 2j - 1 + t$ (where $0 \le t \le 1$), then ratio chasing yields $d(A_1A_iA_m) = \frac{1}{2}\left(1 - \frac{t}{t+2j-1}\frac{1-t}{(1-t)+(2M+1-2j)}\right)$. For fixed $t$, this quantity is minimized when $(t+2j-1)((1-t)+(2M+1-2j))$, a quadratic in $j$ with negative leading coefficient, is as small as possible. But concave functions are minimized on the boundary, so we can assume by symmetry that $j = 1$. Finally, the desired minimum is

$$\min_{t\in[0,1]} \frac{1}{2}\left(1 - \frac{t(1-t)}{(1+t)(N-1-t)}\right) = \frac{\left(1 + \sqrt{\frac{(N-1)(N-2)}{2}}\right)^2}{N^2},$$

so our answer is $p + q = 1006 \cdot 2011 + 2013 = 2025079$ (this is a quadratic problem, so calculus is not necessary).

**Comment 1.** One can do a lot of preliminary smoothing on $P$ (which in fact works when 2013 is replaced by a positive even integer—try it!), but in the end the convexity/weighted averages idea seems essential.

**Comment 2.** There is a nice way (by Lewis Chen) to find $\min_{t\in[0,1]} \frac{1}{2}\left(1 - \frac{t(1-t)}{(1+t)(N-1-t)}\right)$. Indeed, if $a = 1 + t$ and $b = N - 1 - t$, then this quantity simply becomes

$$\frac{1}{N}\left(\frac{1}{a} + \frac{\frac{(N-1)(N-2)}{2}}{b}\right) \ge \frac{1}{N(a+b)}\left(1 + \sqrt{\frac{(N-1)(N-2)}{2}}\right)^2 = \frac{1}{N^2}\left(1 + \sqrt{\frac{(N-1)(N-2)}{2}}\right)^2$$

by Titu's lemma (there is also a direct interpretation for this expression), with equality at

$$b = a\sqrt{\frac{(N-1)(N-2)}{2}} \implies a = \frac{N}{1 + \sqrt{\frac{(N-1)(N-2)}{2}}} \in (1, 2),$$

which is valid (as then $t \in (0,1)$).

This problem was proposed by Victor Wang.

30. Let $P(x)$ denote the polynomial

$$3\sum_{k=0}^{9} x^k + 2\sum_{k=10}^{1209} x^k + \sum_{k=1210}^{146409} x^k.$$

Find the smallest positive integer $n$ for which there exist polynomials $f, g$ with integer coefficients satisfying $x^n - 1 = (x^{16}+1)P(x)f(x) + 11 \cdot g(x)$.

**Answer:** $\boxed{35431200}$.

**Solution.** Let $p = 11$ and $m = 10$; then we can define $Q(x) = (x-1)P(x) = x^{mp^4} + x^{mp^2} + x^m - 3$.

Working in $\mathbb{F}_p$, we see that $Q'(1) = m \neq 0$, so 1 is not a double root of $Q$ and $P(1) \neq 0$ (this can also be verified directly). Thus $P(x) \mid x^n - 1$ iff $P(x) \mid \frac{x^n-1}{x-1}$ or equivalently, $Q(x) \mid x^n - 1$.

We now show that for positive integers $n$, (i) $Q(x) \mid x^n - 1$ (in $\mathbb{F}_p$) iff $m(p^6-1) \mid n$ and (ii) $x^{16}+1 \mid x^n - 1$ iff $32 \mid n$. The "if" direction is easy since $Q(x) \mid Q(x)^{p^2} - Q(x) = Q(x^{p^2}) - Q(x) = x^{mp^6} - x^m$ (and $Q(0) \neq 0$) and $x^{16}+1 \mid x^{32} - 1$. The "only if" direction for (ii) is also easy. (Why?)

For the "only if" direction of (i), first observe that $Q(x) \mid x^n - 1 \implies m \mid n$. (Why?) If $n = mN$, then we find $f(x) = x^{p^4} + x^{p^2} + x - 3 \mid x^N - 1$. From the "if" direction, it will be enough to consider the case $0 < N \leq p^6 - 1$. (Why?) Let $N = p^4 A + B$, where $0 \leq A \leq p^2 - 1$ and $0 \leq B \leq p^4 - 1$. If we set

$$S(x) = (3 - x - x^{p^2})^A x^B - 1;$$

then

$$0 \equiv x^N - 1 \equiv (3 - x - x^{p^2})^A x^B - 1 = S(x) \pmod{f(x)}.$$

But since $(A, B) \neq (0,0)$, $p^2 A + B \geq \deg f = p^4$ (otherwise we would need $(3 - x - x^{p^2})^A x^B = 1$). In particular, $A, B > 0$. Now write

$$0 \equiv S(x) = g(x)x^{p^4} + h(x) \equiv (3 - x - x^{p^2})g(x) + h(x) \pmod{f(x)}$$

for polynomials $g, h$ with $\deg h < p^4$; observe that since $B < p^4$, $[x^B]h(x) = [x^B]S(x) = 3^A \neq 0$. Clearly

$$\deg g = \deg S - p^4 = p^2 A + B - p^4,$$

so $\deg{(3 - x - x^{p^2})g(x)} \leq p^2 + p^2(p^2-1) + (p^4-1) - p^4 = p^4 - 1$ and thus $(3 - x - x^{p^2})g(x) + h(x) = 0$ (in $\mathbb{F}_p$). Yet from earlier we have $[x^B]h(x) \neq 0$, so $[x^B](3 - x - x^{p^2})g(x) \neq 0$ and then $p^2 + p^2 A + B - p^4 \geq B$ forces $A = p^2 - 1$. Hence

$$\begin{aligned} 0 \equiv (3 - x - x^{p^2})S(x) &= (3 - x - x^{p^2})^{A+1} x^B - (3 - x - x^{p^2}) \\ &= (3 - x^{p^2} - x^{p^4})x^B - (3 - x - x^{p^2}) \equiv x^{B+1} + x^{p^2} + x - 3 \pmod{f(x)}, \end{aligned}$$

whence $B = p^4 - 1$. Along with $A = p^2 - 1$ this implies $N = p^6 - 1$, as desired.

Combining (i) and (ii), we must have $2m(p^6-1) = \operatorname{lcm}(32, m(p^6-1)) \mid n$ (it's not difficult to find $v_2(m(p^6-1)) = 4$). On the other hand, $P(x) \mid Q(x) \mid x^{m(p^6-1)} - 1$ while $x^{16}+1 \mid x^{m(p^6-1)}+1$, so $(x^{16}+1)P(x) \mid x^{2m(p^6-1)} - 1$ and the desired answer is $2m(p^6-1) = 35431200$.

**Solution 2.** Let $f(x) = x^{p^4} + x^{p^2} + x - 3$ as in the previous solution; here we present another proof of the crucial fact that $f(x) \mid x^N - 1$ if and only if $p^6 - 1 \mid N$. Since $p \neq 3$, $f(x) \mid x^N - 1$ is equivalent to $x^{p^4} + x^{p^2} + x - 1 \mid x^N - 3^{-N}$ by Fermat's little theorem (consider $f(3x)$).

Now define the sequence $\{c_i\}_{i=-\infty}^{\infty}$ by $c_i = 0$ for $i < 0$, $c_0 = 1$, and for $i > 0$, $c_i = c_{i-1} + c_{i-p^2} + c_{i-p^4}$. Then in fact $f(x) \mid x^N - 1$ if and only if $3^{-N}c_N \equiv 1 \pmod p$ and $c_{N-1} \equiv c_{N-2} \equiv \cdots \equiv c_{N-p^4+1} \equiv 0 \pmod p$. (**) (Why?)

Observe that $\binom{a+b+c}{a,b,c} \equiv 0 \pmod p$ whenever $a+b+c \geq p^2$ and $0 \leq a, b, c < p^2$, e.g. since if $a > 0$, then $\binom{a+b+c}{a,b,c} = \frac{a+b+c}{a}\binom{a+b+c-1}{a-1,b,c}$ and $p^2 \nmid a$. (***)

**Lemma.** Let $i$ be a nonnegative integer less than $p^6 - 1$. If $i + 1 = \overline{d_2 d_1 d_0}$ in base $p^2$ (possibly with leading zeros), then

$$c_i \equiv \binom{d_0 + d_1 + d_2}{d_0, d_1, d_2} = \frac{(d_0+d_1+d_2)!}{d_0! d_1! d_2!} \pmod p.$$

*Proof Outline.* We proceed by strong induction on $i \geq 0$, where the base case $i = 0$ is clear. Now suppose $1 \leq i \leq p^6 - 2$ and assume the result up to $i - 1$. The inductive step then follows from generalization of Pascal's identity and (***), since $c_i = c_{i-1} + c_{i-p^2} + c_{i-p^4}$. (There are two possible issues to consider: (i) leading zeros among $d_1, d_2$ and (ii) carries—we leave the details to the interested reader, but note that (***) is quite important.)$\blacksquare$

By (***) and the lemma, we find that $c_{p^6-2} \equiv c_{p^6-3} \equiv \cdots \equiv c_{p^6-p^4} \equiv 0 \pmod p$, so

$$c_{p^6-1} = c_{p^6-2} + c_{p^6-p^2-1} + c_{p^6-p^4-1} \equiv 0 + 0 + \binom{p^2-1+0+0}{p^2-1, 0, 0} = 1 \pmod p.$$

But we also get $c_{(d_2+1)(p^4-p^2)-1} \equiv \binom{d_2+(p^2-1-d_2)+0}{d_2, p^2-1-d_2, 0} \not\equiv 0 \pmod p$ for $d_2 = 0, 1, \ldots, p^2 - 1$ (e.g. since $[t^{d_2}](1+t)^{p^2-1} \equiv [t^{d_2}]\frac{1+t^{p^2}}{1+t} \not\equiv 0 \pmod p$), so we see that $p^6 - 1$ is the smallest $N$ such that $c_{N-1} \equiv c_{N-2} \equiv \cdots \equiv c_{N-p^4+1} \equiv 0 \pmod p$. But $3^{-(p^6-1)}c_{p^6-1} \equiv c_{p^6-1} \equiv 1 \pmod p$ by Fermat's little theorem, so by (**) $p^6 - 1$ is in fact the smallest $N$ such that $f(x) \mid x^N - 1$, whence $f(x) \mid x^N - 1$ if and only if $p^6 - 1 \mid N$ (in terms of the $c_i$, this follows from the fact that $c_i \equiv c_{i+p^6-1} \pmod p$ for all $i \geq 0$, which in turn follows from $c_i \equiv c_{i+p^6-1} \pmod p$ for $i \in [-(p^4-1), 0]$).

**Comment.** This problem was inspired by POTD 9/13 from WOOT 2010-2011 (restricted access): "Define the sequence $a_n = n$ for $n = 0, 1, \ldots 4$ and $a_n = a_{n-1} + a_{n-5}$ for $n \geq 5$. Suppose that $r_1, r_2, r_3$ are the respective remainders when $a_{50}, a_{100}, a_{150}$ are divided by 5. Find $100 \cdot r_1 + 10 \cdot r_2 + r_3$." Both solutions above, particularly the second (with its "magical formula"), can be motivated by this simpler case (replacing $x^{p^4} + x^{p^2} + x - 1$ by $x^{p^2} + x^p + x - 1$ and then by $x^p + x - 1$). Indeed, the "2-digit" analog of $x^p + x - 1$ allows some sort of finite difference/degree considerations, which might not be as clear in the "3-digit" case.

For instance, one solution to the simpler version goes as follows: we work modulo $p$. To prove that the period of the sequence divides $p^2 - 1$, it suffices to find two consecutive, congruent $p$-tuplets. Notice that

$$(a_{-p}, a_{-p+1}, \ldots, a_{-2}, a_{-1}) = (p-1, 1, 1, \ldots, 1, 1)$$

(simply extending the definition of the $a_i$ to negative $i$). By double induction on $k$ and then $i$ with Pascal's identity, we can prove that for $1 \le k \le p-2$ and $0 \le i \le p-1$,

$$a_{kp+i} \equiv \binom{k+i}{k+1} - \binom{k+i}{k-1} \pmod{p}.$$

(This can be found by experimenting in groups of $p$; it's sort of motivated by finite differences.) It then follows through some simple evaluation that

$$(a_{p^2-p-1}, a_{p^2-p}, \ldots, a_{p^2-3}, a_{p^2-2}) \equiv (p-1, 1, 1, \ldots, 1, 1).$$

This problem was proposed by Victor Wang. The second solution was provided by Brian Chen.

# Acknowledgments


## Contest Directors

Ray Li, James Tao, Victor Wang

## Head Problem Writers

Ray Li, Victor Wang

## Problem Contributors

Ray Li, James Tao, Anderson Wang, Victor Wang, David Yang, Alex Zhu

## Proofreaders and Test Solvers

Mitchell Lee, James Tao, Anderson Wang, David Yang, Alex Zhu

## Website Manager

Ray Li


Online Math Open