The Online Math Open Fall Contest
September 24-October 1, 2012

Contest Information

Format

The test will start Monday September 24 and end Monday October 1. You will have until 7pm EST on October 1 to submit your answers. The test consists of 30 short answer questions, each of which has a nonnegative integer answer. The problem difficulties range from those of AMC problems to those of Olympiad problems. Problems are ordered in roughly increasing order of difficulty.

Team Guidelines

Students may compete in teams of up to four people. Participating students must not have graduated from high school. International students may participate. No student can be a part of more than one team. The members of each team do not get individual accounts; they will all share the team account.

Each team will submit its final answers through its team account. Though teams can save drafts for their answers, the current interface does not allow for much flexibility in communication between team members. We recommend using Google Docs and Spreadsheets to discuss problems and compare answers, especially if teammates cannot communicate in person. Teams may spend as much time as they like on the test before the deadline.

Aids

Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids are not allowed. This is includes (but is not limited to) Geogebra and graphing calculators. **Published print and electronic resources are not permitted.** (This is a change from last year's rules.)

Four-function calculators are permitted on the Online Math Open. That is, calculators which perform only the four basic arithmetic operations (+-*/) may be used. Any other computational aids such as scientific and graphing calculators, computer programs and applications such as Mathematica, and online databases is prohibited. All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.

Clarifications

Clarifications will be posted as they are answered. For the Fall 2012-2013 Contest, they will be posted at here. If you have a question about a problem, please email OnlineMathOpenTeam@gmail.com with "Clarification" in the subject. We have the right to deny clarification requests that we feel we cannot answer.

Scoring

Each problem will be worth one point. Ties will be broken based on the highest problem number that a team answered correctly. If there are still ties, those will be broken by the second highest problem solved, and so on.

Results

After the contest is over, we will release the answers to the problems within the next day. If you have a protest about an answer, you may send an email to OnlineMathOpenTeam@gmail.com (Include "Protest" in the subject). Solutions and results will be released in the following weeks.

1. Calvin was asked to evaluate $37 + 31 \times a$ for some number a. Unfortunately, his paper was tilted 45 degrees, so he mistook multiplication for addition (and vice versa) and evaluated $37 \times 31 + a$ instead. Fortunately, Calvin still arrived at the correct answer while still following the order of operations. For what value of a could this have happened?

2. Petya gave Vasya a number puzzle. Petya chose a digit X and said, "I am thinking of a number that is divisible by 11. The hundreds digit is X and the tens digit is 3. Find the units digit." Vasya was excited because he knew how to solve this problem, but then realized that the problem Petya gave did not have an answer. What digit X did Petya chose?

3. Darwin takes an 11×11 grid of lattice points and connects every pair of points that are 1 unit apart, creating a 10×10 grid of unit squares. If he never retraced any segment, what is the total length of all segments that he drew?

4. Let $\operatorname{lcm}(a, b)$ denote the least common multiple of a and b. Find the sum of all positive integers x such that $x \leq 100$ and $\operatorname{lcm}(16, x) = 16x$.

5. Two circles have radius 5 and 26. The smaller circle passes through center of the larger one. What is the difference between the lengths of the longest and shortest chords of the larger circle that are tangent to the smaller circle?

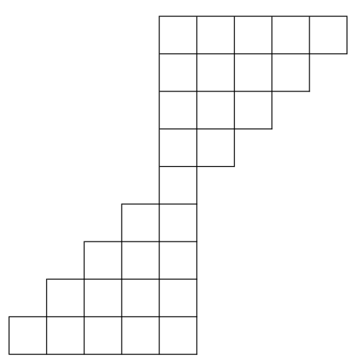
6. An elephant writes a sequence of numbers on a board starting with 1. Each minute, it doubles the sum of all the numbers on the board so far, and without erasing anything, writes the result on the board. It stops after writing a number greater than one billion. How many distinct prime factors does the largest number on the board have?

7. Two distinct points A and B are chosen at random from 15 points equally spaced around a circle centered at O such that each pair of points A and B has the same probability of being chosen. The probability that the perpendicular bisectors of OA and OB intersect strictly inside the circle can be expressed in the form $\frac{m}{n}$, where m, n are relatively prime positive integers. Find $m + n$.

8. In triangle ABC let D be the foot of the altitude from A. Suppose that $AD = 4$, $BD = 3$, $CD = 2$, and AB is extended past B to a point E such that $BE = 5$. Determine the value of CE^2.

9. Define a sequence of integers by $T_1 = 2$ and for $n \geq 2$, $T_n = 2^{T_{n-1}}$. Find the remainder when $T_1 + T_2 + \cdots + T_{256}$ is divided by 255.

10. There are 29 unit squares in the diagram below. A frog starts in one of the five (unit) squares on the top row. Each second, it hops either to the square directly below its current square (if that square exists), or to the square down one unit and left one unit of its current square (if that square exists), until it reaches the bottom. Before it reaches the bottom, it must make a hop every second. How many distinct paths (from the top row to the bottom row) can the frog take?



11. Let $ABCD$ be a rectangle. Circles with diameters AB and CD meet at points P and Q inside the rectangle such that P is closer to segment BC than Q. Let M and N be the midpoints of segments AB and CD. If $\angle MPN = 40^\circ$, find the degree measure of $\angle BPC$.

12. Let $a_1, a_2, \ldots$ be a sequence defined by $a_1 = 1$ and for $n \geq 1$, $a_{n+1} = \sqrt{a_n^2 - 2a_n + 3} + 1$. Find a_{513}.

13. A number is called *6-composite* if it has exactly 6 composite factors. What is the 6th smallest 6-composite number? (A number is *composite* if it has a factor not equal to 1 or itself. In particular, 1 is not composite.)

14. When Applejack begins to buck trees, she starts off with 100 energy. Every minute, she may either choose to buck n trees and lose 1 energy, where n is her current energy, or rest (i.e. buck 0 trees) and gain 1 energy. What is the maximum number of trees she can buck after 60 minutes have passed?

15. How many sequences of nonnegative integers $a_1, a_2, \ldots, a_n$ $(n \geq 1)$ are there such that $a_1 \cdot a_n > 0$, $a_1 + a_2 + \cdots + a_n = 10$, and $\displaystyle\prod_{i=1}^{n-1}(a_i + a_{i+1}) > 0$?

16. Let ABC be a triangle with $AB = 4024$, $AC = 4024$, and $BC = 2012$. The reflection of line AC over line AB meets the circumcircle of $\triangle ABC$ at a point $D \neq A$. Find the length of segment CD.

17. Find the number of integers a with $1 \leq a \leq 2012$ for which there exist nonnegative integers x, y, z satisfying the equation
$$x^2(x^2 + 2z) - y^2(y^2 + 2z) = a.$$

18. There are 32 people at a conference. Initially nobody at the conference knows the name of anyone else. The conference holds several 16-person meetings in succession, in which each person at the meeting learns (or relearns) the name of the other fifteen people. What is the minimum number of meetings needed until every person knows everyone elses name?

19. In trapezoid $ABCD$, $AB < CD$, $AB \perp BC$, $AB \parallel CD$, and the diagonals AC, BD are perpendicular at point P. There is a point Q on ray CA past A such that $QD \perp DC$. If
$$\frac{QP}{AP} + \frac{AP}{QP} = \left(\frac{51}{14}\right)^4 - 2,$$
then $\dfrac{BP}{AP} - \dfrac{AP}{BP}$ can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m, n. Compute $m + n$.

20. The numbers $1, 2, \ldots, 2012$ are written on a blackboard. Each minute, a student goes up to the board, chooses two numbers x and y, erases them, and writes the number $2x + 2y$ on the board. This continues until only one number N remains. Find the remainder when the maximum possible value of N is divided by 1000.

21. A game is played with 16 cards laid out in a row. Each card has a black side and a red side, and initially the face-up sides of the cards alternate black and red with the leftmost card black-side-up. A move consists of taking a consecutive sequence of cards (possibly only containing 1 card) with leftmost

card black-side-up and the rest of the cards red-side-up, and flipping all of these cards over. The game ends when a move can no longer be made. What is the maximum possible number of moves that can be made before the game ends?

22. Let $c_1, c_2, \ldots, c_{6030}$ be 6030 real numbers. Suppose that for any 6030 real numbers $a_1, a_2, \ldots, a_{6030}$, there exist 6030 real numbers $\{b_1, b_2, \ldots, b_{6030}\}$ such that
$$a_n = \sum_{k=1}^{n} b_{\gcd(k,n)}$$
and
$$b_n = \sum_{d|n} c_d a_{n/d}$$
for $n = 1, 2, \ldots, 6030$. Find c_{6030}.

23. For reals $x \geq 3$, let $f(x)$ denote the function
$$f(x) = \frac{-x + x\sqrt{4x-3}}{2}.$$
Let $a_1, a_2, \ldots,$ be the sequence satisfying $a_1 > 3$, $a_{2013} = 2013$, and for $n = 1, 2, \ldots, 2012$, $a_{n+1} = f(a_n)$. Determine the value of
$$a_1 + \sum_{i=1}^{2012} \frac{a_{i+1}^3}{a_i^2 + a_i a_{i+1} + a_{i+1}^2}.$$

24. In scalene $\triangle ABC$, I is the incenter, I_a is the A-excenter, D is the midpoint of arc BC of the circumcircle of ABC, and M is the midpoint of side BC. Extend ray IM past M to point P such that $IM = MP$. Let Q be the intersection of DP and MI_a, and R be the point on the line MI_a such that $AR \parallel DP$. Given that $\frac{AI_a}{AI} = 9$, the ratio $\frac{QM}{RI_a}$ can be expressed in the form $\frac{m}{n}$ for two relatively prime positive integers m, n. Compute $m + n$.

25. Suppose 2012 reals are selected independently and at random from the unit interval $[0, 1]$, and then written in nondecreasing order as $x_1 \leq x_2 \leq \cdots \leq x_{2012}$. If the probability that $x_{i+1} - x_i \leq \frac{1}{2011}$ for $i = 1, 2, \ldots, 2011$ can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m, n, find the remainder when $m + n$ is divided by 1000.

26. Find the smallest positive integer k such that
$$\binom{x+kb}{12} \equiv \binom{x}{12} \pmod{b}$$
for all positive integers b and x. (Note: For integers a, b, c we say $a \equiv b \pmod{c}$ if and only if $a - b$ is divisible by c.)

27. Let ABC be a triangle with circumcircle ω. Let the bisector of $\angle ABC$ meet segment AC at D and circle ω at $M \neq B$. The circumcircle of $\triangle BDC$ meets line AB at $E \neq B$, and CE meets ω at $P \neq C$. The bisector of $\angle PMC$ meets segment AC at $Q \neq C$. Given that $PQ = MC$, determine the degree measure of $\angle ABC$.

28. Find the remainder when
$$\sum_{k=1}^{2^{16}} \binom{2k}{k} (3 \cdot 2^{14} + 1)^k (k-1)^{2^{16}-1}$$
is divided by $2^{16} + 1$. (*Note:* It is well-known that $2^{16} + 1 = 65537$ is prime.)

29. In the Cartesian plane, let $S_{i,j} = \{(x, y) \mid i \le x \le j\}$. For $i = 0, 1, \ldots, 2012$, color $S_{i,i+1}$ pink if i is even and gray if i is odd. For a convex polygon P in the plane, let $d(P)$ denote its pink density, i.e. the fraction of its total area that is pink. Call a polygon P *pinxtreme* if it lies completely in the region $S_{0,2013}$ and has at least one vertex on each of the lines $x = 0$ and $x = 2013$. Given that the minimum value of $d(P)$ over all non-degenerate convex pinxtreme polygons P in the plane can be expressed in the form $\frac{(1+\sqrt{p})^2}{q^2}$ for positive integers p, q, find $p + q$.

30. Let $P(x)$ denote the polynomial
$$3 \sum_{k=0}^{9} x^k + 2 \sum_{k=10}^{1209} x^k + \sum_{k=1210}^{146409} x^k.$$
Find the smallest positive integer n for which there exist polynomials f, g with integer coefficients satisfying $x^n - 1 = (x^{16} + 1)P(x)f(x) + 11 \cdot g(x)$.

Acknowledgments

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