4^{th} **National Internet Mathematical Olympiad**

Official Solutions

Winter Contest NIMO 2013

March 1 – March 11, 2013

1. Find the remainder when $2+4+\cdots+2014$ is divided by $1+3+\cdots+2013$. Justify your answer.

   Proposed by Evan Chen

   Solution. Let $A = 2+4+\cdots+2014$ and $B = 1+3+\cdots+2013$. Then

   $$A - B = \underbrace{1+1+\cdots+1}_{1007\ 1's} = 1007.$$

   Other solutions may explicitly compute $A = 1007 \cdot 1008$ and $B = 1007^2$ to arrive at the same conclusion. Since $B > 1007$, the remainder is 1007. □

2. Square $\mathcal{S}$ has vertices $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$. Points P and Q are independently selected, uniformly at random, from the perimeter of $\mathcal{S}$. Determine, with proof, the probability that the slope of line PQ is positive.

   Proposed by Isabella Grabski

   Solution. The answer is $\frac{1}{2}$. Let $A = (1,0)$, $B = (0,1)$, $C = (-1,0)$ and $D = (0,-1)$. Consider a point $P \in \overline{CD}$ (the other cases are analogous). We claim that even for a fixed P, the probability is $\frac{1}{2}$.

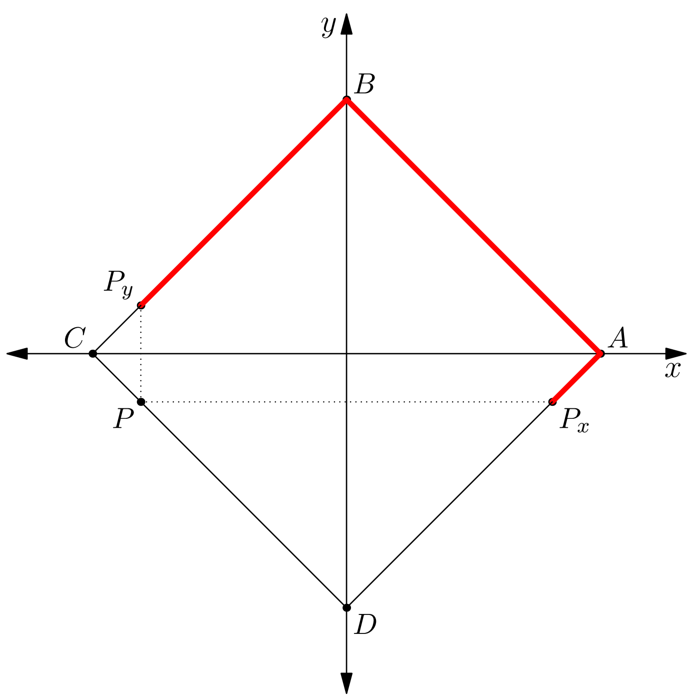

Figure 1: Desired region for problem 2, highlighted in red.

Let P_x be the point on $\overline{AD}$ with PP_x parallel to the x-axis, and define P_y analogously. Then it is easy to see that the region where Q can lie is precisely the polygonal line P_yBAP_x.

Since $P_yC = PC = P_xA$, it easy to see that the desired region is precisely one-half the perimeter of $\mathcal{S}$. Hence the claim holds, and the answer is therefore $\frac{1}{2}$. □

Solution 2. We will show the answer is $\frac{1}{2}$. Remark that the probability that a line has either undefined or zero slope is 0. Therefore it suffices to consider lines with either positive or negative slopes.

Notice that by reflecting a line PQ over the x-axis we obtain a line $P'Q'$ whose slope is the negative of line PQ. We may ignore the cases where the slope is 0 or not defined. Hence, by symmetry, the answer is $\frac{1}{2}$. □

Note: It's important to note (although no points will be deducted for failing to do so) that symmetry is very different from having a $1-1$ correspondence. The real importance is to note that reflection preserves the length of segments; it is not sufficient to find a bijection from lines with positive slope to lines with negative slope, because the sets involved are infinite.

For the sake of example, consider the problem:

> x is chosen randomly in the interval $[0,1)$. What is the probability that $x \leq \frac{1}{3}$?

It is easy to construct a bijection between $\left[0, \frac{1}{3}\right)$ and $\left[\frac{1}{3}, 1\right)$; namely, $x \mapsto 2x + \frac{1}{3}$. But this certainly does not imply the answer is $\frac{1}{2}$!

When we say we pick a point "uniformly and at random from the perimeter of $\mathcal{S}$", this is equivalent to saying that the probability a point is selected from some interval is proportional to the length of that interval. That's why the length-preserving property (which may be expressed more perversely as just "symmetry") plays a crucial role in this problem.

3. Let ABC be a triangle. Prove that there exists a unique point P for which one can find points D, E and F such that the quadrilaterals $APBF$, $BPCD$, $CPAE$, $EPFA$, $FPDB$, and $DPEC$ are all parallelograms.

   Proposed by Lewis Chen

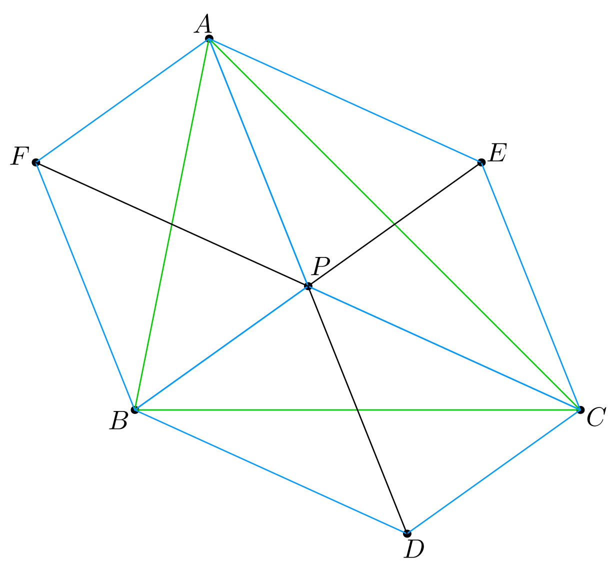

Figure 2: Parallelograms

Solution. We claim that the centroid of $\triangle ABC$ is the unique point with the property.

First, suppose P, D, E, F have the desired property. Clearly, $\overline{CP} \parallel \overline{BD}$ and $\overline{FP} \parallel \overline{PD}$. This implies that C, P, F are collinear. Similarly, A, P, D and B, P, E are collinear.

Now, $\overline{FP}$ bisects $\overline{AB}$ because $APBF$ is a parallelogram. Hence $\overline{CP}$ is a median of $\triangle ABC$. Similarly, $\overline{AP}$ and $\overline{BP}$ are medians, so P must be the centroid of $\triangle ABC$.

Conversely, if P is the centroid, then select D, E, F to be the reflections of P over the midpoints of sides BC, CA, AB; this forces $APBF$, $BPCD$ and $CPAE$ to be parallelograms. Then $\overline{BP} \parallel \overline{FA}$ and $\overline{CP} \parallel \overline{EA}$, so $AFPE$ is a parallelogram. Similarly, $BDPF$ and $CEPD$ are parallelograms, as desired. □

Solution 2. The solution is by vectors. Note the $WXYZ$ is a parallelogram if and only if

$$\overrightarrow{W} + \overrightarrow{Y} = \overrightarrow{X} + \overrightarrow{Z}.$$

The condition is equivalent to finding $\overrightarrow{P}$ for which there exists $\overrightarrow{D}, \overrightarrow{E}, \overrightarrow{F}$ such that

$$\begin{aligned}
\overrightarrow{A} + \overrightarrow{B} &= \overrightarrow{P} + \overrightarrow{F} \\
\overrightarrow{B} + \overrightarrow{C} &= \overrightarrow{P} + \overrightarrow{D} \\
\overrightarrow{C} + \overrightarrow{A} &= \overrightarrow{P} + \overrightarrow{E} \\
\overrightarrow{E} + \overrightarrow{F} &= \overrightarrow{P} + \overrightarrow{A} \\
\overrightarrow{F} + \overrightarrow{E} &= \overrightarrow{P} + \overrightarrow{B} \\
\overrightarrow{D} + \overrightarrow{D} &= \overrightarrow{P} + \overrightarrow{C}
\end{aligned}$$

Adding twice the sum of the first three equations with the following three, we obtain that

$$\overrightarrow{P} = \frac{\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C}}{3}$$

i.e. P is the centroid of $\triangle ABC$. It is easy to verify this works. □

4. Let $\mathcal{F}$ be the set of all 2013×2013 arrays whose entries are 0 and 1. A transformation $K : \mathcal{F} \to \mathcal{F}$ is defined as follows: for each entry a_{ij} in an array $A \in \mathcal{F}$, let S_{ij} denote the sum of all the entries of A sharing either a row or column (or both) with a_{ij}. Then a_{ij} is replaced by the remainder when S_{ij} is divided by two.

   Prove that for any $A \in \mathcal{F}$, $K(A) = K(K(A))$.

   Proposed by Aaron Lin

Solution. We will show the result holds for any $n \times n$ grid, where n is odd.

Let the entries of A be a_{ij}, $1 \le i, j \le n$. Let the entries of $K(A)$ be b_{ij}, and let the entries of $K(K(A))$ be c_{ij}.

By symmetry, it suffices to check that $c_{11} = b_{11}$. Now, let $R = a_{11} + a_{12} + \cdots + a_{1n}$ and $C = a_{11} + a_{21} + \cdots + a_{n1}$. Then,

$$\begin{aligned}
c_{11} &\equiv -b_{11} + \sum_{k=1}^{n} b_{k1} + \sum_{k=1}^{n} b_{1k} \pmod 2 \\
&= b_{11} + \sum_{k=2}^{n} b_{k1} + \sum_{k=2}^{n} b_{1k} \\
&= b_{11} + \sum_{k=2}^{n} \left(C + \sum_{j=2}^{n} a_{kj} \right) + \sum_{k=2}^{n} \left(R + \sum_{j=2}^{n} a_{jk} \right) \\
&= b_{11} + \sum_{k=2}^{n} \sum_{j=2}^{n} a_{kj} + \sum_{k=2}^{n} \sum_{j=2}^{n} a_{jk} + (n-1)C + (n-1)R \\
&= b_{11} + 2 \sum_{2 \le x, y \le n} a_{xy} + (n-1)C + (n-1)R \\
&\equiv b_{11} \pmod 2
\end{aligned}$$

which implies $c_{11} = b_{11}$ as desired. □

5. In convex hexagon $AXBYCZ$, sides AX, BY and CZ are parallel to diagonals BC, XC and XY, respectively. Prove that $\triangle ABC$ and $\triangle XYZ$ have the same area.

   Proposed by Evan Chen

   Solution. Let $[\mathcal{P}]$ denote the area of a polygon $\mathcal{P}$.

   The important claim is that if $\overline{KL} \parallel \overline{MN}$, then $[KLM] = [KLN]$. This is a simple consequence of the formula $A = \frac{1}{2}bh$.

   Then, we find that

$$\begin{aligned} [ABC] &= [XBC] \quad (\text{since } \overline{AX} \parallel \overline{BC}) \\ &= [XYC] \quad (\text{since } \overline{BY} \parallel \overline{XC}) \\ &= [XYZ] \quad (\text{since } \overline{CZ} \parallel \overline{XY}) \end{aligned}$$

   as desired. □

6. A strictly increasing sequence $\{x_i\}_{i=1}^{\infty}$ of positive integers is said to be *large* if, for every real number L, there exists an integer n such that $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} > L$. Do there exist large sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ such that the sequence $\{a_i + b_i\}_{i=1}^{\infty}$ is not large?

   Proposed by Lewis Chen

   Solution. The answer is yes. We present one of many possible constructions.

   The idea is to select a_i and b_i such that $a_i + b_i = 2^i$, which will make $\{a_i + b_i\}_{i=1}^{\infty}$ not large.

   Let the a_i and b_i alternate in runs, where one sequence increases by one at each step, as shown in the table below.

i	1	2	3	4	5	6	$\cdots$	16	17	18	$2^{16}-8$	
a_i	1	2	3	4	19	50	$\cdots$	$2^{16}-24$	$2^{16}-23$	$\cdots$	$2^{17}-48$	$\cdots$
b_i	1	2	5	12	13	14	$\cdots$	24	$2^{16}+23$	$\cdots$	$2^{2^{16}-8}-2^{17}+48$	$\cdots$
							length 12			length $2^{16}-24$		

Table 1: a_i and b_i

   Formally, we define the sequences by $t_1 = 2$, $a_1 = b_1 = 1$, $a_2 = b_2 = 2$, and

$$t_{n+1} = t_n + \max\{a_{t_n}, b_{t_n}\}$$

$$a_{n+1} = \begin{cases} a_n + 1 & \text{if } \exists k : t_{2k-1} + 1 \le n \le t_{2k} \\ 2^{n+1} - b_n & \text{otherwise} \end{cases}$$

$$b_{n+1} = \begin{cases} b_n + 1 & \text{if } \exists k : t_{2k} + 1 \le n \le t_{2k+1} \\ 2^{n+1} - a_n & \text{otherwise.} \end{cases}$$

   Note that

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \underbrace{\frac{1}{2n} + \cdots + \frac{1}{2n}}_{n \text{ terms}} = \frac{1}{2}.$$

   This construction guarantees that

$$\sum_{i=t_{2k-1}+1}^{t_{2k}} \frac{1}{a_i} > \frac{1}{2}.$$

   Similar equations hold for b_i. Therefore $\{a_i\}$ and $\{b_i\}$ are large. So we're done. □

7. Let a, b, c be positive reals satisfying $a^3 + b^3 + c^3 + abc = 4$. Prove that

$$\frac{(5a^2+bc)^2}{(a+b)(a+c)} + \frac{(5b^2+ca)^2}{(b+c)(b+a)} + \frac{(5c^2+ab)^2}{(c+a)(c+b)} \geq \frac{(a^3+b^3+c^3+6)^2}{a+b+c}$$

and determine the cases of equality.

Proposed by Evan Chen

Solution. The equality cases are $a = b = c = 1$ and the cyclic permutations of

$$(a, b, c) = \left(\frac{2}{\sqrt[3]{3}}, \frac{1}{\sqrt[3]{3}}, \frac{1}{\sqrt[3]{3}} \right).$$

By the Cauchy-Schwarz inequality,

$$\sum_{\text{cyc}} \frac{(5a^3+abc)^2}{a^2(a+b)(a+c)} \geq \frac{\left(\sum_{\text{cyc}} 5a^3 + abc\right)^2}{\sum_{\text{cyc}} a^2(a+b)(a+c)} = \frac{\left(5a^3+5b^3+5c^3+3abc\right)^2}{(a+b+c)(a^3+b^3+c^3+abc)}$$

which simplifies to the desired right-hand side.

Equality occurs if and only if

$$\frac{5a^3+abc}{a^2(a+b)(a+c)} = \frac{5b^3+abc}{b^2(b+c)(b+a)} = \frac{5c^3+abc}{c^2(c+a)(c+b)}$$

Multiplying by $abc(a+b)(b+c)(c+a)$ we observe this is equivalent to

$$bc(5a^2+bc)(b+c) = ca(5b^2+ca)(c+a) = ab(5c^2+ab)(a+b)$$

Let us assume without loss of generality that $c \geq \max\{a, b\}$. We now find that

$$\begin{aligned} 0 &= b(5a^2+bc)(b+c) - a(5b^2+ac)(a+c) \\ &= 5abc(a-b) + c(b^3-a^3) + c^2(b^2-a^2) \\ &= c(b-a)(-5ab + a^2 + ab + b^2 + c(a+b)) \\ &= c(b-a)(a^2+b^2+c(a+b)-4ab) \end{aligned}$$

But,

$$a^2 + b^2 + ca + cb - 4ab \geq 2(a^2+b^2) - 4ab \geq 0$$

with equality only when $a = b = c$. This forces $a = b$; otherwise the two factors are both nonzero. Now, if we set $t = \frac{c}{a} = \frac{c}{b}$ we find that

$$0 = t(5+t)(t+1) - (5t^2+1)(2) = t^3 - 4t^2 + 5t - 2 = (t-2)(t-1)^2$$

which gives the equality cases claimed above. □

8. For a finite set X define

$$S(X) = \sum_{x \in X} x \text{ and } P(x) = \prod_{x \in X} x.$$

Let A and B be two finite sets of positive integers such that $|A| = |B|$, $P(A) = P(B)$ and $S(A) \neq S(B)$. Suppose for any $n \in A \cup B$ and prime p dividing n, we have $p^{36} \mid n$ and $p^{37} \nmid n$. Prove that

$$|S(A) - S(B)| > 1.9 \cdot 10^6.$$

Proposed by Evan Chen

Solution. Let $A = \left\{a_1^{36}, a_2^{36}, \cdots, a_n^{36}\right\}$ and $B = \left\{b_1^{36}, b_2^{36}, \cdots, b_n^{36}\right\}$. Notice that $a_1 a_2 \cdots a_n = b_1 b_2 \cdots b_n$ and the a_i, b_i are squarefree.

The crucial component of the solution is the claim

Claim. For any prime p such that $p - 1 \mid 36$, we have $S(A) \equiv S(B) \pmod{p}$.

Proof. Let $A_p = \{a \in A \mid p \text{ divides } a\}$ and define B_p analogously. The condition that the a_i and b_i are squarefree, together with $P(A) = P(B)$, imply that $|A_p| = |B_p|$. Now by Fermat's Little Theorem, we see that

$$a^{p-1} \equiv \begin{cases} 1 & a \not\equiv 0 \pmod{p} \\ 0 & a \equiv 0 \pmod{p} \end{cases}.$$

So $S(A) \equiv n - |A_p| \pmod{p}, S(B) \equiv n - |B_p| \pmod{p} \implies S(A) \equiv S(B) \pmod{p}$. ■

Now $S(A) - S(B) \equiv 0 \pmod{p}$ for $p \in \{2, 3, 5, 7, 13, 19, 37\}$. Hence, $S(A) - S(B)$ is divisible by

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 = 1919190 > 1.9 \cdot 10^6$$

which implies the conclusion upon remarking that $S(A) - S(B) \neq 0$. □