NIMO Winter Contest 2013

1. This is an eight-problem exam. Your team of up to four will be allotted 60 minutes to complete the exam.

2. You must show work and justify your reasoning to receive credit. Answers without justification will receive little or no credit. Conversely, a well-written solution with a few minor errors may receive near-full or full credit.

3. No aids other that scratch paper, graph paper, rulers, compasses, and protractors are permitted. In particular, calculators, slide rules, or other computational aids are not allowed. Use of computers is permitted only for communication among team members and for typing and/or submitting solutions.

4. **For the 2013 contest, solutions may be either typed or handwritten. Note that this is a change from previous years.**

   - If you choose to write your solutions by hand, make sure your solution is legible and dark enough to be processed by a machine - we cannot grade solutions that are unreadable.
   - If you choose to type your solutions, we strongly encourage you to use LaTeX. Note that your solution must still be uploaded as a PDF should you submit it electronically.

   No page should include solutions to more than one problem – even if your solution is less than one page, please start a new page for each problem.

5. You must submit your solutions on or before March 11, 2013. You must submit your solutions by one of the following methods:

   - Mail all of your solutions with cover sheet (details on our website) in order in a single envelope addressed to:

     NIMO
     P.O. Box 988
     Pleasanton, CA 94566

     Solutions must be postmarked on or before March 11, 2013.
   - Upload your solutions in a single PDF file through your team's account at our site. You will be allotted up to 1 hour after the contest to scan and upload your solutions.

6. Your team's results will be posted on the NIMO website when available. Your team captain will also be sent an email when scores are available.

7. **Please read all of the rules on the NIMO website before beginning the exam.**

1. Find the remainder when $2+4+\cdots+2014$ is divided by $1+3+\cdots+2013$. Justify your answer.

2. Square $\mathcal{S}$ has vertices $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$. Points P and Q are independently selected, uniformly at random, from the perimeter of $\mathcal{S}$. Determine, with proof, the probability that the slope of line PQ is positive.

3. Let ABC be a triangle. Prove that there exists a unique point P for which one can find points D, E and F such that the quadrilaterals $APBF$, $BPCD$, $CPAE$, $EPFA$, $FPDB$, and $DPEC$ are all parallelograms.

4. Let $\mathcal{F}$ be the set of all 2013×2013 arrays whose entries are 0 and 1. A transformation $K:\mathcal{F}\to\mathcal{F}$ is defined as follows: for each entry a_{ij} in an array $A\in\mathcal{F}$, let S_{ij} denote the sum of all the entries of A sharing either a row or column (or both) with a_{ij}. Then a_{ij} is replaced by the remainder when S_{ij} is divided by two.

   Prove that for any $A\in\mathcal{F}$, $K(A)=K(K(A))$.

5. In convex hexagon $AXBYCZ$, sides AX, BY and CZ are parallel to diagonals BC, XC and XY, respectively. Prove that $\triangle ABC$ and $\triangle XYZ$ have the same area.

6. A strictly increasing sequence $\{x_i\}_{i=1}^{\infty}$ of positive integers is said to be *large* if, for every real number L, there exists an integer n such that $\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}>L$. Do there exist large sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ such that the sequence $\{a_i+b_i\}_{i=1}^{\infty}$ is not large?

7. Let a,b,c be positive reals satisfying $a^3+b^3+c^3+abc=4$. Prove that

$$\frac{(5a^2+bc)^2}{(a+b)(a+c)}+\frac{(5b^2+ca)^2}{(b+c)(b+a)}+\frac{(5c^2+ab)^2}{(c+a)(c+b)}\geq\frac{(a^3+b^3+c^3+6)^2}{a+b+c}$$

   and determine the cases of equality.

8. For a finite set X define

$$S(X)=\sum_{x\in X}x \text{ and } P(x)=\prod_{x\in X}x.$$

   Let A and B be two finite sets of positive integers such that $|A|=|B|$, $P(A)=P(B)$ and $S(A)\neq S(B)$. Suppose for any $n\in A\cup B$ and prime p dividing n, we have $p^{36}\mid n$ and $p^{37}\nmid n$. Prove that

$$|S(A)-S(B)|>1.9\cdot 10^6.$$