Question 11: The USA Yes/No Olympiad

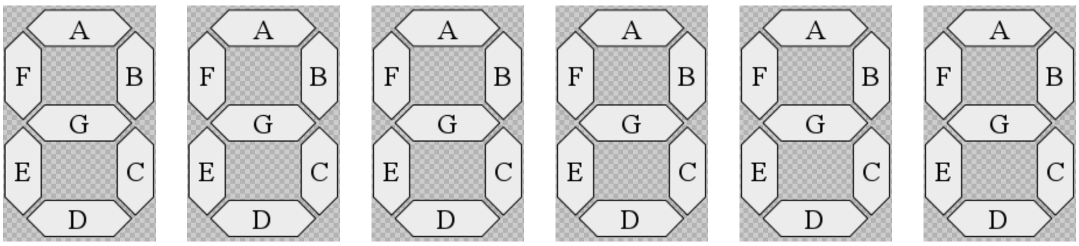
NIMO 2013 April Fools' Contest

April 1, 2013

Welcome to the first USA Yes/No Olympiad! Here we have compiled several yes/no problems from around the world. There are six problems, each worth seven marks.

When you solve a problem, fill in the corresponding hexagon below if and only if the answer to that problem is "yes". When all problems have been solved, the figure below will reveal the final answer.

This is the 11th problem of the NIMO April Fools' Contest. The remainder of the contest can be found at `http://www.internetolympiad.org`.



Problem 6 Problem 5 Problem 4 Problem 3 Problem 2 Problem 1

1^{st} United States of America Yes/No Olympiad

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Dai I   April 1, 2013

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USAYNO 1. (a) Is $2^{70} + 3^{70}$ divisible by 13?

(b) Let p be a prime for which $(p-1)^p + 1$ is a power of p. Does it follow that $p \leq 100$?

(c) Let S be the set of all positive integers of the form $19a + 85b$, where $a, b \in \mathbb{Z}^+$. On the real axis, the points of S are colored in red and the remaining integer numbers are colored in green. Is there a point A on the real axis so that any two points with integer coordinates which are symmetrical with respect to A have necessarily distinct colors?

(d) Does there exist an integer $n \geq 100$ for which $n \mid 2^n + 2$?

(e) Is it possible to pair off the numbers $1, 2, \ldots, 20$ so that the sum of each of the ten pairs are ten different prime numbers?

(f) Is there a base b in which all numbers $10101_b, 101010101_b, 1010101010101_b, \ldots$ are primes?

(g) Is it possible to fit 106 spheres with unit diameter in a $10 \times 10 \times 1$ box?

USAYNO 2. (a) Does there exist a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x + f(y)) = f(x) - y$ for all $x, y \in \mathbb{Z}$?

(b) Does there exist a row of Pascal's triangle which contains four distinct elements u, v, x, y satisfying $x = 2u$ and $y = 2v$?

(c) Every point in the plane is colored either cyan or turquoise. Can this be done such that, for any point O, there are exactly two cyan points P for which $OP = 2002$?

(d) Every point in the plane is colored either cyan or turquoise. Can this be done such that, for any point O, there is exactly one cyan point P for which $OP = 2002$?

(e) Do there exist infinitely many integers n for which such that $n \mid 2^{n-1} + 3^{n-1}$?

(f) A 9×12 rectangle is partitioned into unit squares. The centers of all the squares, except for the four corner squares and the eight squares orthogonally adjacent to them, are colored in red. Determine if it possible to draw a closed broken line such that (a) a point is a vertex of the broken line if and only if it is the center of a red square, (b) each edge of the broken line has length $\sqrt{13}$, and (c) the broken line exhibits central symmetry.

(g) Does the equation $x^4 = y^2 + z^2 + 4$ have integer solutions?

USAYNO 3. (a) Let $S = \{a^3 + b^3 + c^3 - 3abc \mid a, b, c \in \mathbb{Z}\}$. Is S closed under multiplication?

(b) Are there functions $g, h : \mathbb{R} \to \mathbb{R}$ such that there is exactly one function $f : \mathbb{R} \to \mathbb{R}$ satisfying $g \circ f = f \circ g$ and $h \circ f = f \circ h$?

(c) Can every bijection from $\mathbb{Z}$ to $\mathbb{Z}$ be written as the sum of two such bijections?

(d) Does there exist a set S of integers for which $S \subseteq S + S$, and 0 is the only integer which cannot be expressed as the sum of the elements of a finite nonempty subset of S?

(e) Given only 1×4 and 4×1 rectangles, is it possible to tile a 13×13 table from which the central square is removed?

(f) Do there exist infinitely many integers $n \geq 0$ such that $\gcd(n, 10) = 1$ and $s(n^3) = s(n) + s(n^2)$, where $s(k)$ denotes the sum of the base-10 digits of k?

(g) Let $f(n)$ denote the largest prime factor of n. Do there exist infinitely many sets of distinct integers a, b, c such that $f(a^2 + 1) = f(b^2 + 1) = f(c^2 + 1)$?

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Dai II April 1, 2013

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USAYNO 4. (a) Do there exist integers $a, n \geq 1$ such that $1 + \frac{1}{1+a} + \frac{1}{1+2a} + \cdots + \frac{1}{1+na}$ is an integer?

(b) Are there positive integers a, b, c, d for which $a^4 + b^4 + c^4 + d^4$ is a fourth power?

(c) Do there exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x))$ is strictly increasing but $g(f(x))$ is strictly decreasing?

(d) Is there a prime $p \leq 100$ for which no integers x, y, z satisfy $y^{37} + pz = x^3 + 11$?

(e) Are there infinitely many triples of relatively prime positive integers a, b, c such that $\frac{bc+b+c}{a}$, $\frac{ca+c+a}{b}$ and $\frac{ab+a+b}{c}$ are all integers?

(f) Are there integers $a, b, c \geq 42$ so that $a^2 + 2b + c$, $b^2 + 2c + a$ and $c^2 + 2a + b$ are squares?

(g) Is there a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $\left(19^{f(n)} - 97\right)/2^n \in \mathbb{Z}$ for each $n \in \mathbb{Z}$?

USAYNO 5. (a) Can one find 4004 integers so that the sum of any 2003 of them is not divisible by 2003?

(b) Each vertex of a finite graph can be colored either black or white. Initially all vertices are black. We are allowed to pick a vertex P and change the color of P and all of its neighbors. Is it possible to change every vertex from black to white by a sequence of such moves?

(c) Does there exist a function $f : \mathbb{Z} \to \mathbb{Z}$ for which $f(f(n)) = 3n$ for all n?

(d) Is there a sequence $a_1, a_2, \ldots$ of positive reals such that for all n, $a_1 + a_2 + \cdots + a_n \leq n^2$ and $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \leq 2008$?

(e) Do there exist positive integers x, y for which $3y^2 = x^4 + x$?

(f) Do there exist 16 three digit numbers, using only three different digits in all, so that all the numbers leave different residues when divided by 16?

(g) Does there exist a partition of the positive integers into 12 sets such that for any positive integers n and $0 < i < j < 13$, the numbers in and jn are in different sets?

USAYNO 6. (a) Are there infinitely many integers $m \neq n$ for which $m^m + n$ divides $n^n + m$?

(b) Is it true that $\sum_{i=1}^{100} \frac{a_i}{a_{i+1}+a_{i+2}} \geq 50$ holds for any positive reals $a_1, a_2, \cdots, a_{100}$? Here $a_{101} = a_1$ and $a_{102} = a_2$.

(c) Is there a polynomial $P(x, y) \in \mathbb{R}[x, y]$ such that $P(\mathbb{Z}^+, \mathbb{Z}^+) \cap \mathbb{Z}^+$ is precisely the set of positive integers which are not triangular numbers?

(d) Let S be a set of points in the plane such that no three points are collinear, and the distance between any two points is rational. Must S be finite?

(e) Fix a line ℓ and consider an infinite set E of congruent ellipses in the plane. Suppose that every line parallel to ℓ intersects at least one ellipse in E. Does it follow that there are infinitely many triples of ellipses in E with a line passing through all three?

(f) If $\omega^n = 1$ $(n \in \mathbb{Z}^+)$ and $\text{Re}[\omega]$ is an algebraic integer, does it follow that $\omega^{2008} = 1$?

(g) Does there exist an algebraic integer with absolute value 1 which is not a root of unity?