NIMO Monthly Contest

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Contest 7   8:00 PM – 8:40 PM EDT

May 27, 2013

1. At ARML, Santa is asked to give rubber duckies to 2013 students, one for each student. The students are conveniently numbered $1, 2, \cdots, 2013$, and for any integers $1 \le m < n \le 2013$, students m and n are friends if and only if $0 \le n - 2m \le 1$.

   Santa has only four different colors of duckies, but because he wants each student to feel special, he decides to give duckies of different colors to any two students who are either friends or who share a common friend. Let N denote the number of ways in which he can select a color for each student. Find the remainder when N is divided by 1000.

2. In $\triangle ABC$, points E and F lie on $\overline{AC}, \overline{AB}$, respectively. Denote by P the intersection of $\overline{BE}$ and $\overline{CF}$. Compute the maximum possible area of $\triangle ABC$ if $PB = 14$, $PC = 4$, $PE = 7$, $PF = 2$.

3. Richard has a four infinitely large piles of coins: a pile of pennies (worth 1 cent each), a pile of nickels (5 cents), a pile of dimes (10 cents), and a pile of quarters (25 cents). He chooses one pile at random and takes one coin from that pile. Richard then repeats this process until the sum of the values of the coins he has taken is an integer number of dollars. (One dollar is 100 cents.) What is the expected value of this final sum of money, in cents?

4. Find the positive integer N for which there exist reals $\alpha, \beta, \gamma, \theta$ which obey

$$\begin{aligned} 0.1 &= \sin\gamma\cos\theta\sin\alpha, \\ 0.2 &= \sin\gamma\sin\theta\cos\alpha, \\ 0.3 &= \cos\gamma\cos\theta\sin\beta, \\ 0.4 &= \cos\gamma\sin\theta\cos\beta, \\ 0.5 &\ge \left|N - 100\cos 2\theta\right|. \end{aligned}$$

5. For every integer $n \ge 1$, the function $f_n : \{0, 1, \cdots, n\} \to \mathbb{R}$ is defined recursively by $f_n(0) = 0$, $f_n(1) = 1$ and

$$(n-k)f_n(k-1) + kf_n(k+1) = nf_n(k)$$

   for each $1 \le k < n$. Let $S_N = f_{N+1}(1) + f_{N+2}(2) + \cdots + f_{2N}(N)$. Find the remainder when $\lfloor S_{2013} \rfloor$ is divided by 2011. (Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.)