NIMO Monthly Contest

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Contest 6 8:00 PM – 8:45 PM EST

February 24, 2013

1. Find the sum of all primes that can be written both as a sum of two primes and as a difference of two primes.

2. Let f be a function from positive integers to positive integers where $f(n) = \frac{n}{2}$ if n is even and $f(n) = 3n+1$ if n is odd. If a is the smallest positive integer satisfying
$$\underbrace{f(f(\cdots f}_{2013\ f\text{'s}}(a)\cdots)) = 2013,$$
find the remainder when a is divided by 1000.

3. Find the integer $n \geq 48$ for which the number of trailing zeros in the decimal representation of $n!$ is exactly $n-48$.

4. While taking the SAT, you become distracted by your own answer sheet. Because you are not bound to the College Board's limiting rules, you realize that there are actually 32 ways to mark your answer for each question, because you could fight the system and bubble in multiple letters at once: for example, you could mark AB, or AC, or ABD, or even $ABCDE$, or nothing at all!

   You begin to wonder how many ways you could mark off the 10 questions you haven't yet answered. To increase the challenge, you wonder how many ways you could mark off the rest of your answer sheet without ever marking the same letter twice in a row. (For example, if ABD is marked for one question, AC cannot be marked for the next one because A would be marked twice in a row.) If the number of ways to do this can be expressed in the form $2^m p^n$, where $m, n > 1$ are integers and p is a prime, compute $100m + n + p$.

5. Zang is at the point $(3,3)$ in the coordinate plane. Every second, he can move one unit up or one unit right, but he may never visit points where the x and y coordinates are both composite. In how many ways can he reach the point $(20,13)$?

6. For each positive integer n, let $H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$. If
$$\sum_{n=4}^{\infty} \frac{1}{nH_nH_{n-1}} = \frac{M}{N}$$
for relatively prime positive integers M and N, compute $100M + N$.

7. In $\triangle ABC$ with $AB = 10$, $AC = 13$, and $\measuredangle ABC = 30^\circ$, M is the midpoint of $\overline{BC}$ and the circle with diameter $\overline{AM}$ meets $\overline{CB}$ and $\overline{CA}$ again at D and E, respectively. The area of $\triangle DEM$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Compute $100m + n$.

8. Find the number of positive integers n for which there exists a sequence $x_1, x_2, \cdots, x_n$ of integers with the following property: if indices $1 \leq i \leq j \leq n$ satisfy $i + j \leq n$ and $x_i - x_j$ is divisible by 3, then $x_{i+j} + x_i + x_j + 1$ is divisible by 3.

9. Let $ABCD$ be a square of side length 6. Points E and F are selected on rays AB and AD such that segments EF and BC intersect at a point L, D lies between A and F, and the area of $\triangle AEF$ is 36. Clio constructs triangle PQR with $PQ = BL$, $QR = CL$ and $RP = DF$, and notices that the area of $\triangle PQR$ is $\sqrt{6}$. If the sum of all possible values of DF is $\sqrt{m} + \sqrt{n}$ for positive integers $m \geq n$, compute $100m + n$.

10. Let $x \neq y$ be positive reals satisfying $x^3 + 2013y = y^3 + 2013x$, and let $M = \left(\sqrt{3}+1\right)x + 2y$. Determine the maximum possible value of M^2.

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