NIMO Winter 2012 Solutions Manual

1. We are looking for the total number of distinct slopes of the lines connecting two points on the grid. This is equal to twice the number of positive slopes, plus two (to account for vertical and horizontal lines).

   Because slope is equal to $\frac{\Delta y}{\Delta x}$, all positive slopes can be written in the form $\frac{a}{b}$ for $1 \leq a, b \leq 9$. Each ordered pair (a, b) produces a distinct slope unless $\gcd(a, b) > 1$. By listing, we find that there are 55 distinct positive slopes. Thus, there are a total of $2(55) + 2 = \boxed{112}$ different slopes, and thus a maximum of 112 non-parallel lines can be drawn.

2. By Vieta's Formulas, $r_1 + r_2 + r_3 = 5$ and $r_1r_2 + r_2r_3 + r_3r_1 = 6$. Thus, $r_1^2 + r_2^2 + r_3^2 = (r_1 + r_2 + r_3)^2 - 2(r_1r_2 + r_2r_3 + r_3r_1) = 5^2 - 2(6) = \boxed{13}$.

3. Regardless of sign, each odd number contributes an odd amount to the sum and each even number contributes an even amount to the sum. Because there are 1006 odd numbers, the expression must evaluate to an even number, so all odd remainders cannot be achieved.

   Now, we show that each even number can be achieved. For all $1 \leq i \leq 1006$, take the sum $1 + 2 + 3 + \cdots + (i-1) - i + (i+1) + \cdots + 2012$. This sum is equal to $\frac{2012 \cdot 2013}{2} - 2i \equiv 1006 - 2i \pmod{2012}$. But $503 - i$ takes all values from 1 to 1006 modulo 1006 for $1 \leq i \leq 1006$, so $1006 - 2i$ takes on all even values modulo 2012. Hence, all even remainders can be achieved, so the answer is $\boxed{1006}$.

4. Each set of four points $A_aA_bB_cB_d$ defines exactly one intersection point, and each intersection point is defined by a set of four points $A_aA_bB_cB_d$. Thus, there are $\binom{n}{2}\binom{n+1}{2}$ total intersections between these segments. This is equal to $\frac{(n-1)n^2(n+1)}{4}$. Because $n-1, n, n+1$ are three consecutive integers, one of them must be divisible by 3. It follows that $Q = \frac{(n-1)n^2(n+1)}{4}$ is divisible by 3.

5. Assume $\overline{AF}$, $\overline{BC}$, and $\overline{DE}$ are pairwise nonparallel. Denote the intersection of $\overleftrightarrow{AF}$ and $\overleftrightarrow{BC}$ by X, the intersection of $\overleftrightarrow{BC}$ and $\overleftrightarrow{DE}$ by Y, and the intersection of $\overleftrightarrow{DE}$ and $\overleftrightarrow{AF}$ by Z. Because $\angle A \cong \angle B$, it follows that $\angle XAB \cong \angle XBA$ and thus $\triangle XAB$ is isosceles. Then, the perpendicular bisector of $\overline{AB}$ is the angle bisector of $\angle AXB = \angle ZXY$. Similarly, the perpendicular bisectors of $\overline{CD}$ and $\overline{EF}$ are the angle bisectors of $\angle XYZ$ and $\angle YZX$, respectively. The angle bisectors of $\angle ZXY$, $\angle XYZ$, and $\angle YZX$ are concurrent at the incenter of $\triangle XYZ$, so the perpendicular bisectors of $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ are concurrent in this case.

   If two of the sides are parallel, then assume, without loss of generality, that $\overline{AF} \parallel \overline{BC}$. Define Y and Z as they were defined in the previous part. Because $\overline{AZ} \parallel \overline{BY}$, quadrilateral $ABYZ$ is a trapezoid. Again, the perpendicular bisectors of $\overline{CD}$ and $\overline{EF}$ are the angle bisectors of $\angle BYZ$ and $\angle YZA$, respectively. Denote by P the midpoint of $\overline{YZ}$ and by Q the intersection of the angle bisectors of $\angle BYZ$ and $\angle YZA$. Now, because $m\angle AZY + m\angle BYZ = 180^\circ$, we have $m\angle QZY + m\angle QYZ = 90^\circ$, from which $\angle YQZ$ is right. It follows that Q lies on the circle with diameter $\overline{YZ}$, so $PY = PZ = PQ$. Thus, $\angle PQY \cong \angle QYP$, and because $\overline{YQ}$ bisects $\angle BYZ$, we have $\angle PQY \cong \angle BYQ$. Hence, $\overline{PQ} \parallel \overline{BY}$. Because P is the midpoint of $\overline{YZ}$, $\overline{PQ}$ must be the midline of trapezoid $ABYZ$. But the perpendicular bisector of $\overline{AB}$ is also the midline of trapezoid $ABYZ$, so Q lies on the perpendicular bisector of $\overline{AB}$ as desired.

6. Let $x = a(a+b+c)$ and $y = bc$. Then, $x + y = (a+b)(a+c)$. The given equation implies:

$$\begin{aligned} 4xy &= (x+y)^2 \\ x^2 - 2xy + y^2 &= 0 \\ (x-y)^2 &= 0 \\ x &= y. \end{aligned}$$

   Thus, $a(a+b+c) = bc$ as desired.

7. By Legendre's Theorem, $v_p(n!) = \frac{n-s_p(n)}{p-1}$, where p is a prime, $v_p(n)$ is the exponent of the prime p that divides n, and $s_p(n)$ is the sum of the digits of n when written in base p.

   Choosing $p = 2$ yields $v_2(n!) = n - s_2(n)$. The given condition holds iff $v_2(n!) = n - s_2(n) \geq n - 2$, or $s_2(n) \leq 2$. Thus, the sum of the digits of n in base 2 is 1 or 2. There are 9 positive integers n not greater than 500 such that the sum of the digits in the binary representation of n is 1, and 36 positive integers n not greater than 500 such that the sum of the digits in the binary representation of n is 2. The answer is $9 + 36 = \boxed{45}$.

   The answers 44 and 45 were both accepted for this problem, as the case $n = 1$ can be argued to be either valid or invalid.

8. **Lemma 1:** $\angle A_{i-1}A_iA_{i+1} \cong \angle A_{i+1005}A_{i+1006}A_{i+1007}$.

   Proof: Consider the partition through $A_{i+503}A_{i+1509}$. Regardless of whether the two resultant polygons are rotations or reflections of each other, $\angle A_{i-1}A_iA_{i+1}$ and $\angle A_{i+1005}A_{i+1006}A_{i+1007}$ are opposite $\overline{A_{i+503}A_{i+1509}}$, because if it is a reflection, then $\angle A_{i-1}A_iA_{i+1} \cong \angle A_{i+1007}A_{i+1006}A_{i+1005}$, and if it is a rotation, then $\angle A_{i-1}A_iA_{i+1} \cong m < A_{i+1005}A_{i+1006}A_{i+1007}$.

Lemma 2: $A_iA_{i+1} = A_{i+1006}A_{i+1007}$.

Proof: For the sake of contradiction, suppose the contrary; that is, suppose that $A_iA_{i+1} \neq A_{i+1006}A_{i+1007}$. Then, consider the partitioning line $\overline{A_{i+1}A_{i+1007}}$. It follows that the two resultant polygons must be reflections of each other, because our assumption is contradicted if they are rotations. Thus, $A_iA_{i+1} = A_{i+1}A_{i+2}$, and $A_{i+1006}A_{i+1007} = A_{i+1007}A_{i+1008}$. In particular, $A_{i+1}A_{i+2} \neq A_{i+1007}A_{i+1008}$.

In general, from $A_kA_{k+1} \neq A_{k+1006}A_{k+1007}$, consider the partitioning line $\overline{A_{k+1}A_{k+1007}}$ to conclude that $A_kA_{k+1} = A_{k+1}A_{k+2}$, $A_{k+1006}A_{k+1007} = A_{k+1007}A_{k+1008}$, and thus $A_{k+1}A_{k+2} \neq A_{k+1007}A_{k+1008}$.

After repeating this argument 1006 times, we may conclude that $A_iA_{i+1} = A_{i+1}A_{i+2} = A_{i+2}A_{i+3} = \cdots = A_{i+1006}A_{1007}$, contradicting our assumption. Hence, the initial assumption must be false, and the lemma must be true.

Lemma 3: $A_iA_j = A_{i+1006}A_{j+1006}$.

Proof: Consider the polygons $A_iA_{i+1}A_{i+2}\dots A_j$ and $A_{i+1006}A_{i+1007}A_{i+1008}\dots A_{j+1006}$. By Lemma 2, they share at least all but one congruent corresponding side length. By Lemma 1, they share all congruent corresponding angles. Hence, they are congruent polygons, so the lemma is true.

By Lemma 3, $A_iA_j = A_{i+1006}A_{j+1006}$ and $A_jA_{i+1006} = A_{j+1006}A_i$, so $A_iA_jA_{i+1006}A_{j+1006}$ is a parallelogram, as desired.