NIMO Winter Contest 2012

1. This is an eight-problem exam. Your team of up to four will be allotted 60 minutes to complete the exam.

2. You must show work and justify your reasoning to receive credit. Answers without justification will receive little or no credit. Conversely, a well-written solution with a few minor errors may receive near-full or full credit.

3. No aids other that scratch paper, graph paper, rulers, compasses, and protractors are permitted. In particular, calculators, slide rules, or other computational aids are not allowed. Use of computers is permitted only for communication among team members and for submitting solutions.

4. All solutions must be handwritten on the provided answer sheets. Please make sure your handwriting is legible and dark enough to be processed by a machine - we cannot grade solutions that are unreadable. No page should include solutions to more than one problem.

5. You must submit your solutions on or before January 28, 2012. You must submit your solutions by one of the following methods:

   (a) Mail all of your solutions with cover sheet (details on our website) in order in a single envelope addressed to:

   NIMO
   P.O. Box 988
   Pleasanton, CA 94566

   Solutions must be postmarked on or before January 28, 2012.

   (b) Upload your solutions in a single PDF file through your team's account at our site. You will be allotted up to 1 hour after the contest to scan and upload your solutions.

6. Your team's results will be posted on the NIMO website when available. Your team captain will also be sent an email when scores are available.

7. **Please read all of the rules on the NIMO website before beginning the exam.**

The Problems

1. In a 10 by 10 grid of dots, what is the maximum number of lines that can be drawn connecting two dots on the grid so that no two lines are parallel?

2. If r_1, r_2, and r_3 are the solutions to the equation $x^3 - 5x^2 + 6x - 1 = 0$, then what is the value of $r_1^2 + r_2^2 + r_2^3$?

3. The expression $\circ 1 \circ 2 \circ 3 \circ \cdots \circ 2012$ is written on a blackboard. Catherine places a $+$ sign or a $-$ sign into each blank. She then evaluates the expression, and finds the remainder when it is divided by 2012. How many possible values are there for this remainder?

4. Parallel lines ℓ_1 and ℓ_2 are drawn in a plane. Points $A_1, A_2, \ldots, A_n$ are chosen on ℓ_1, and points $B_1, B_2, \ldots, B_{n+1}$ are chosen on ℓ_2. All segments A_iB_j are drawn, such that $1 \leq i \leq n$ and $1 \leq j \leq n+1$. Let the number of total intersections between these segments (not including endpoints) be denoted by Q. Given that no three segments are concurrent, besides at endpoints, prove that Q is divisible by 3.

5. In convex hexagon $ABCDEF$, $\angle A \cong \angle B$, $\angle C \cong \angle D$, and $\angle E \cong \angle F$. Prove that the perpendicular bisectors of $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ pass through a common point.

6. The positive numbers a, b, c satisfy $4abc(a + b + c) = (a + b)^2(a + c)^2$. Prove that $a(a + b + c) = bc$.

7. For how many positive integers $n \leq 500$ is $n!$ divisible by 2^{n-2}?

8. A convex 2012-gon $A_1A_2A_3 \ldots A_{2012}$ has the property that for every integer $1 \leq i \leq 1006$, $\overline{A_iA_{i+1006}}$ partitions the polygon into two congruent regions. Show that for every pair of integers $1 \leq j < k \leq 1006$, quadrilateral $A_jA_kA_{j+1006}A_{k+1006}$ is a parallelogram.