NIMO Summer Contest 2012 Solutions

1. Let $f(x) = (x^4 + 2x^3 + 4x^2 + 2x + 1)^5$. Compute the prime p satisfying $f(p) = 418{,}195{,}493$.

   Answer: $\boxed{2}$

   Solution 1: Since p is prime, either $p = 2$ or $p \equiv 1 \pmod 2$. If $p \equiv 1 \pmod 2$, then

   $$418{,}195{,}493 = f(p) = (p^4 + 2p^3 + 4p^2 + 2p + 1)^5 \equiv (1 + 2 + 4 + 2 + 1)^5 \equiv 0 \pmod 2,$$

   a contradiction. Hence, $p = 2$.

   Solution 2: Let $g(x) = x^4 + 2x^3 + 4x^2 + 2x + 1$. It follows that

   $$\begin{aligned} f(p) = (g(p))^5 &= 418{,}195{,}493 \\ &< 10^{10} \\ \implies g(p) &< 10^2. \end{aligned}$$

   For all $x \geq 3$, $g(x) > 10^2$. Hence, $p < 3$, so $p = 2$.

2. Compute the number of positive integers n satisfying the inequalities

   $$2^{n-1} < 5^{n-3} < 3^n.$$

   Answer: $\boxed{5}$

   Solution: Note that

   $$\begin{aligned} 2^{4-1} &> 5^{4-3}, \\ 2^{5-1} &< 5^{5-3}, \end{aligned}$$

   so by induction, the left inequality is satisfied for all integers $n \geq 5$. Also,

   $$\begin{aligned} 5^{9-3} &< 3^9, \\ 5^{10-3} &> 3^{10}, \end{aligned}$$

   so by induction, the right inequality is satisfied for all integers $n \leq 9$. Hence, $5 \leq n \leq 9$, and there are 5 solutions.

3. Let

   $$S = \sum_{i=1}^{2012} i!.$$

   The tens and units digits of S (in decimal notation) are a and b, respectively. Compute $10a + b$.

   Answer: $\boxed{13}$

   Solution: It suffices to find the residue of S modulo 100. Note that

   $$\begin{aligned} S &= \sum_{i=1}^{9} i! + \sum_{i=10}^{2012} i! \\ &= \sum_{i=1}^{9} i! + \sum_{i=10}^{2012} \left(10! \prod_{j=10}^{i} j \right) \\ &\equiv \sum_{i=1}^{9} i! \pmod{100}. \end{aligned}$$

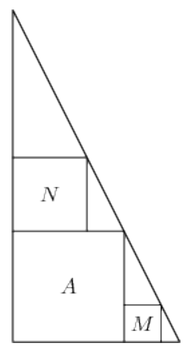
But $1! \equiv 1$, $2! \equiv 2$, $3! \equiv 6$, $4! \equiv 24$, $5! \equiv 20$, $6! \equiv 20$, $7! \equiv 40$, $8! \equiv 20$, $9! \equiv 80$ (where the modulus for all congruences is 100). Thus,

$$S \equiv 13 \pmod{100},$$

and $10a + b = 13$.

4. The degree measures of the angles of nondegenerate hexagon $ABCDEF$ are integers that form an arithmetic sequence in some order, and $\angle A$ is the smallest angle of the (not necessarily convex) hexagon. Compute the sum of all possible degree measures of $\angle A$.

5. In the diagram below, three squares are inscribed in right triangles. Their areas are A, M, and N, as indicated in the diagram. If $M = 5$ and $N = 12$, then A can be expressed as $a + b\sqrt{c}$, where a, b, and c are positive integers and c is not divisible by the square of any prime. Compute $a + b + c$.



6. When Eva counts, she skips all numbers containing a digit divisible by 3. For example, the first ten numbers she counts are 1, 2, 4, 5, 7, 8, 11, 12, 14, 15. What is the 100^{th} number she counts?

7. A permutation $(a_1, a_2, a_3, \ldots, a_{2012})$ of $(1, 2, 3, \ldots, 2012)$ is selected at random. If S is the expected value of

$$\sum_{i=1}^{2012} |a_i - i|,$$

then compute the sum of the prime factors of S.

Answer: $\square$

By linearity of expectation,

$$\begin{aligned} S &= \sum_{i=1}^{2012} \mathbb{E}[|a_i - i|] \\ &= \sum_{i=1}^{2012} \sum_{j=1}^{2012} \mathbb{P}[a_i = j]|j - i| \\ &= \frac{1}{2012} \sum_{i=1}^{2012} \sum_{j=1}^{2012} |j - i| \\ &= TBC \end{aligned}$$

8. Points A, B, and O lie in the plane such that $\angle AOB = 120°$. Circle ω_0 with radius 6 is constructed tangent to both $\overrightarrow{OA}$ and $\overrightarrow{OB}$. For all $i \geq 1$, circle ω_i with radius r_i is constructed such that $r_i < r_{i-1}$ and ω_i is tangent to $\overrightarrow{OA}$, $\overrightarrow{OB}$, and ω_{i-1}. If

$$S = \sum_{i=1}^{\infty} r_i,$$

then S can be expressed as $a\sqrt{b}+c$, where a, b, c are integers and b is not divisible by the square of any prime. Compute $100a+10b+c$.

9. A quadratic polynomial $p(x)$ with integer coefficients satisfies $p(41)=42$. For some integers $a, b>41$, $p(a)=13$ and $p(b)=73$. Compute the value of $p(1)$.

10. A *triangulation* of a polygon is a subdivision of the polygon into triangles meeting edge to edge, with the property that the set of triangle vertices coincides with the set of vertices of the polygon. Adam randomly selects a triangulation of a regular 180-gon. Then, Bob selects one of the 178 triangles in this triangulation. The expected number of 1° angles in this triangle can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a+b$.

    Answer: $\boxed{9089}$

    Solution: Consider the circumcircle of the 180-gon. If $\triangle ABC$ is one of the triangles created in the triangulation, then $\measuredangle BAC=1^\circ$ iff minor arc BC measures 2°; that is, iff $\overline{BC}$ is a side of the polygon. Furthermore, each side of the polygon must be a side of one of the triangles in the triangulation, so each side contributes exactly one 1° angle. Because there are 180 sides of the polygon and 178 triangles to select from, the expected number of 1° angles in the triangle is $\frac{180}{178}=\frac{90}{89}$, so $100a+b=9089$.

11. Let a and b be two positive integers satisfying the equation

$$20\sqrt{12}=a\sqrt{b}.$$

    Compute the sum of all possible distinct products ab.

12. The NEMO (National Electronic Math Olympiad) is similar to the NIMO Summer Contest, in that there are fifteen problems, each worth a set number of points. However, the NEMO is weighted using Fibonacci numbers; that is, the n^{th} problem is worth F_n points, where $F_1=F_2=1$ and $F_n=F_{n-1}+F_{n-2}$ for $n\geq 3$. The two problem writers are fair people, so they make sure that each of them is responsible for problems worth an equal number of total points. Compute the number of ways problem writing assignments can be distributed between the two writers.

13. For the NEMO, Kevin needs to compute the product

$$9\times 99\times 999\times\cdots\times 999999999.$$

    Kevin takes exactly ab seconds to multiply an a-digit integer by a b-digit integer. Compute the minimum number of seconds necessary for Kevin to evaluate the expression together by performing eight such multiplications.

    Answer: $\boxed{870}$

    Solution: It is not difficult to verify that in this problem, the product of an a-digit number and a b-digit number must be an $a+b$-digit number. We will assume this property throughout.

    If the numbers $x_1, x_2, \ldots, x_n$ have $d_1, d_2, \ldots, d_n$ digits, respectively, then the number of seconds required for Kevin to multiply them is exactly

$$\sum_{1\leq i<j\leq n} x_i x_j.$$

    This follows by induction on n. If $n=2$ the claim is obviously true. If $n>2$, then assume x_n is multiplied last (otherwise, reorder the numbers so that this is true). The time required is

$$x_n\left(x_1+x_2+\cdots+x_{n-1}\right)+\sum_{1\leq i<j\leq n-1} x_i x_j=\sum_{1\leq i<j\leq n} x_i x_j,$$

as claimed.

The answer is thus

$$\begin{aligned}\sum_{1\le i<j\le 9} ij &= \frac{1}{2}\left(\sum_{i=1}^{9} i\right)^2 - \frac{1}{2}\sum_{i=1}^{9} i^2 \\ &= \frac{1}{2}\left(\left(\frac{9\cdot 10}{2}\right)^2 - \frac{9\cdot 10\cdot 19}{6}\right) \\ &= \frac{1}{2}\left(45^2 - 285\right) \\ &= 870.\end{aligned}$$

14. A set of lattice points is called *good* if it does not contain two points that form a line with slope -1 or slope 1. Let $S = \{(x, y) \mid x, y \in \mathbb{Z}, 1 \le x, y \le 4\}$. Compute the number of non-empty good subsets of S.

15. In the diagram below, square $ABCD$ with side length 23 is cut into nine rectangles by two lines parallel to $\overline{AB}$ and two lines parallel to $\overline{BC}$. The areas of four of these rectangles are indicated in the diagram. Compute the largest possible value for the area of the central rectangle.