NIMO Contest #4

1. Compute the average of the integers $2, 3, 4, \ldots, 2012$.

2. For which positive integer n is the quantity $\frac{n}{3} + \frac{40}{n}$ minimized?

3. In chess, there are two types of minor pieces, the bishop and the knight. A bishop may move along a diagonal, as long as there are no pieces obstructing its path. A knight may jump to any lattice square $\sqrt{5}$ away as long as it isnt occupied.

   One day, a bishop and a knight were on squares in the same row of an infinite chessboard, when a huge meteor storm occurred, placing a meteor in each square on the chessboard independently and randomly with probability p. Neither the bishop nor the knight were hit, but their movement may have been obstructed by the meteors.

   The value of p that would make the expected number of valid squares that the bishop can move to and the number of squares that the knight can move to equal can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $100a + b$.

4. Let $S = \{(x, y) : x, y \in \{1, 2, 3, \ldots, 2012\}\}$. For all points (a, b), let $N(a, b) = \{(a-1, b), (a+1, b), (a, b-1), (a, b+1)\}$. Kathy constructs a set T by adding n distinct points from S to T at random. If the expected value of $\displaystyle\sum_{(a,b)\in T} |N(a, b) \cap T|$ is 4, then compute n.

5. A number is called *purple* if it can be expressed in the form $\frac{1}{2^a 5^b}$ for positive integers $a > b$. The sum of all purple numbers can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $100a + b$.

6. The polynomial $P(x) = x^3 + \sqrt{6}x^2 - \sqrt{2}x - \sqrt{3}$ has three distinct real roots. Compute the sum of all $0 \le \theta < 360$ such that $P(\tan \theta^\circ) = 0$.

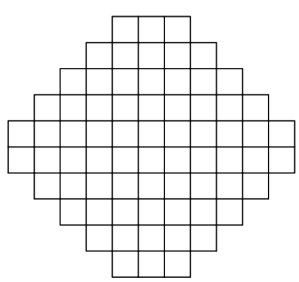
7. In quadrilateral $ABCD$, $AC = BD$ and $\measuredangle B = 60^\circ$. Denote by M and N the midpoints of $\overline{AB}$ and $\overline{CD}$, respectively. If $MN = 12$ and the area of quadrilateral $ABCD$ is 420, then compute AC.

8. Bob has invented the Very Normal Coin (VNC). When the VNC is flipped, it shows heads $\frac{1}{2}$ of the time and tails $\frac{1}{2}$ of the time - unless it has yielded the same result five times in a row, in which case it is guaranteed to yield the opposite result. For example, if Bob flips five heads in a row, then the next flip is guaranteed to be tails.

   Bob flips the VNC an infinite number of times. On the nth flip, Bob bets 2^{-n} dollars that the VNC will show heads (so if the second flip shows heads, Bob wins \$0.25, and if the third flip shows tails, Bob loses \$0.125).

   Assume that dollars are infinitely divisible. Given that the first flip is heads, the expected number of dollars Bob is expected to win can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $100a + b$.

9. In how many ways can the following figure be tiled with 2×1 dominos?



10. In cyclic quadrilateral $ABXC$, $\measuredangle XAB = \measuredangle XAC$. Denote by I the incenter of $\triangle ABC$ and by D the projection of I on $\overline{BC}$. If $AI = 25$, $ID = 7$, and $BC = 14$, then XI can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $100a + b$.