NIMO Monthly Contest 3

1. Hexagon $ABCDEF$ is inscribed in a circle. If $\measuredangle ACE = 35^\circ$ and $\measuredangle CEA = 55^\circ$, then compute the sum of the degree measures of $\angle ABC$ and $\angle EFA$.

2. Compute the number of positive integers $n < 2012$ that share exactly two positive factors with 2012.

3. Compute the sum of the distinct prime factors of 10101.

4. The *subnumbers* of an integer n are the numbers that can be formed by using a contiguous subsequence of the digits. For example, the subnumbers of 135 are 1, 3, 5, 13, 35, and 135. Compute the number of primes less than 1,000,000,000 that have no non-prime subnumbers. One such number is 37, because 3, 7, and 37 are prime, but 135 is not one, because the subnumbers 1, 35, and 135 are not prime.

5. The hour and minute hands on a certain 12-hour analog clock are indistinguishable. If the hands of the clock move continuously, compute the number of times strictly between noon and midnight for which the information on the clock is not sufficient to determine the time.

6. In rhombus $NIMO$, $MN = 150\sqrt{3}$ and $\measuredangle MON = 60^\circ$. Denote by S the locus of points P in the interior of $NIMO$ such that $\angle MPO \cong \angle NPO$. Find the greatest integer not exceeding the perimeter of S.

7. For every pair of reals $0 < a < b < 1$, we define sequences $\{x_n\}_{n\ge 0}$ and $\{y_n\}_{n\ge 0}$ by $x_0 = 0$, $y_0 = 1$, and for each integer $n \ge 1$:

$$x_n = (1-a)x_{n-1} + ay_{n-1},$$
$$y_n = (1-b)x_{n-1} + by_{n-1}.$$

   The *supermean* of a and b is the limit of $\{x_n\}$ as n approaches infinity. Over all pairs of real numbers (p, q) satisfying $\left(p - \frac{1}{2}\right)^2 + \left(q - \frac{1}{2}\right)^2 \le \left(\frac{1}{10}\right)^2$, the minimum possible value of the supermean of p and q can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

8. Concentric circles Ω_1 and Ω_2 with radii 1 and 100, respectively, are drawn with center O. Points A and B are chosen independently at random on the circumferences of Ω_1 and Ω_2, respectively. Denote by ℓ the tangent line to Ω_1 passing through A, and denote by P the reflection of B across ℓ. Compute the expected value of OP^2.

9. Let $f(x) = x^2 - 2x$. A set of real numbers S is *valid* if it satisfies the following:

   (a) If $x \in S$, then $f(x) \in S$.

   (b) If $x \in S$ and $\underbrace{f(f(\dots f}_{k\ f\text{'s}}(x)\dots)) = x$ for some integer k, then $f(x) = x$.

   Compute the number of 7-element valid sets.

10. For reals $x_1, x_2, x_3, \dots, x_{333} \in [-1, \infty)$, let $S_k = \displaystyle\sum_{i=1}^{333} x_i^k$ for each k. If $S_2 = 777$, compute the least possible value of S_3.