NIMO 2010 Solutions Manual

1. Throughout this solution the area of region $\Re$ will be denoted by $[\Re]$.

   Let $O = (0, 0)$, $A = (a, 0)$, $B = (0, b)$, $C = (a, b)$, $X = (x, 0)$, $Y = (0, y)$, $X' = (x, b)$, $Y' = (a, y)$, and $Z = (x, y)$. It follows that $[M] = [OXZY] = xy$. Because $\triangle AXZ \sim \triangle AOB$, we know

   $$\frac{AX}{XZ} = \frac{AO}{OB} \implies \frac{AX}{y} = \frac{a}{b} \implies AX = \frac{ay}{b}$$

   and similarly $BY = \dfrac{bx}{a}$.

   But $AX = ZY'$ and $BY = ZX'$ so

   $$[N] = [ZY'CX'] = ZX' \cdot ZY' = \frac{ay}{b} \cdot \frac{bx}{a} = xy = [M].$$

   Hence, $\dfrac{[M]}{[N]} = \boxed{1}$.

2. If there are common members of the sequences, there exist natural numbers k and l such that

   $$k^2 - 1 = l^2 + 1.$$

   This may be rewritten as

   $$k^2 - l^2 = 2$$
   $$(k + l)(k - l) = 2.$$

   Because k and l are natural numbers, it follows that $k + l = 2$. But then $k - l = 1$, yielding $k = \dfrac{3}{2}$, a contradiction. Hence, no numbers are members of both sequences.

3. Let X be the first meeting point and Y the second. Denote the distance AB by x. Billy and Bobby can meet at at most one point before they reach the point they are traveling to. However, the distance between A and B depends on when they reach the opposite end. We proceed with casework.

   Case 1: Billy and Bobby reach B and A, respectively, before meeting for a second time.
   In this case, Billy travels AX before meeting for the first time while Bobby walks XB. Before they meet for a second time, Billy walks $XB + BY$ and Bobby walks $XA + AY$ in the same time. Because they walk at constant rates, we have:

   $$\frac{AX}{XB} = \frac{XB + BY}{XA + AY}$$
   $$\frac{3}{x - 3} = \frac{x - 3 + 10}{3 + x - 10}$$

   Solving this, we find $x = 0$ or $x = -1$, neither of which can be walking distances in this problem. So there are no solutions in this case.

   Case 2: Billy reaches point B first and meets Bobby at point Y before Bobby reaches A.
   In a similar fashion, we find $x = -12$ or $x = 5$. However, -12 is negative, and 5 is less than 10, so neither can be possible total lengths of the path.

   Case 3: Bobby reaches point A first and meets Billy at point Y before Billy reaches B.
   This yields $x = 4$ or $x = 15$. Because $4 < 10$, the path cannot be 4 units long.
   Thus, the only possible length of the path is $\boxed{15}$ units.

4. We can rewrite the equation as

   $$100(a + d + g) + 10(b + e + h) + (c + f + i) = 1665.$$

Because $100(a+d+g)$ and $10(b+e+h)$ have units digit 0, it follows that $c+f+i$ has units digit 5. But

$$c+f+i \geq 1+2+3 = 6 > 5,$$
$$c+f+i \leq 7+8+9 = 24 < 25,$$

so $c+f+i = 15$. We can again rewrite the equation as

$$100(a+d+g)+10(b+e+h) = 1650$$
$$10(a+d+g)+(b+e+h) = 165.$$

Because $10(a+d+g)$ has units digit 0, it follows that $b+e+h$ has units digit 5. By the same logic as above, we obtain $b+e+h = 15$.

It remains to show that $b+e+h = 15$ is achievable. But this may be achieved by setting $a = 1, b = 2, c = 4, d = 5, e = 6, f = 3, g = 9, h = 7, i = 8$. Hence, the only possible value for $b+e+h$ is $\boxed{15}$.

5. The lightbulbs are located at the points $(\pm 1, \pm 1, \pm 1)$ which determine a cube. Note that a configuration will explode only if both lightbulbs on an edge are simultaneously on.

   We proceed by casework on the number of lightbulbs on.

   If 0 lightbulbs are on then there is only 1 configuration of the lightbulbs, which indeed satisfies the conditions of the problem.

   If 1 lightbulb is on, then there are 8 configurations of the lightbulbs, all of which satisfy the conditions of the problem.

   If 2 lightbulbs are on, then there are $\binom{8}{2} = 28$ configurations of the lightbulbs. However, if both endpoints of an edge are lit then the configuration is not satisfactory so there are $28 - 12 = 16$ solutions for this case.

   To count the number of configurations where 3 lightbulbs are on, first choose a lightbulb to be lit. Notice that if the opposite lightbulb is lit, then no more can be lit; hence, the opposite lightbulb is not lit. Additionally, no lightbulbs that share an edge with the original lightbulb can be lit. So there are 3 lightbulbs left to choose from, and each choice of 2 from these 3 satisfies the conditions, for $\binom{3}{2} = 3$ solutions. But the original lightbulb can be chosen in 8 ways, and each case is counted three times (once for each lightbulb as the starting point), so the number of configurations from this case is $3 \cdot \frac{8}{3} = 8$.

   We count the ways for four lightbulbs to be on in the same way as the previous case. After choosing a starting lightbulb, there is $\binom{3}{3} = 1$ way to light the rest, for a total of $1 \cdot \frac{8}{4} = 2$ configurations.

   In total, there are $1+8+16+8+2 = \boxed{35}$ configurations.

6. Note that $\triangle ACB$ is a right triangle, so CP is the geometric mean of BP and AP. Thus, by the Pythagorean Theorem,

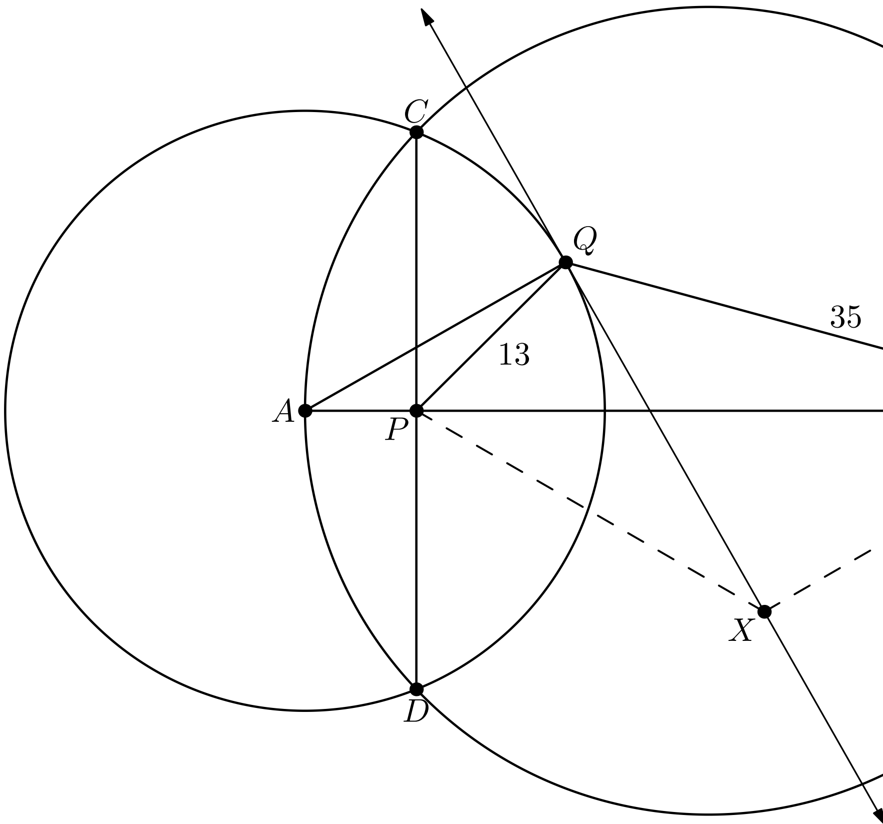
$$AC^2 = CP^2 + AP^2 = (AP \cdot BP) + AP^2 = AP(BP + AP) = AP \cdot AB.$$

   Because $AC = AQ$ it follows that $AQ^2 = AP \cdot AB$. Thus, $\overline{AQ}$ is tangent to the circumcircle of $\triangle BQP$. Since line l is perpendicular to $\overline{AQ}$ through Q, l must pass through the circumcenter of $\triangle BQP$. Hence, X, which is equidistant from points B and P, must be the circumcenter of $\triangle BQP$.

Furthermore, the equation $AP^2 = AP \cdot AB$ implies that $\triangle APQ \sim \triangle AQB$, so $\angle AQP \cong \angle ABQ \cong \angle PBQ$. Thus, $m\angle AQP + m\angle BPQ = m\angle PBQ + m\angle BPQ = 60°$, so $m\angle BQP = 120°$. From the Law of Cosines it follows that $BP = 43$. By the Law of Sines,

$$QX = \frac{BP}{2\sin\angle BQP} = \frac{43}{\sqrt{3}}$$

Hence $QX = \boxed{\frac{43\sqrt{3}}{3}}$.



7. **Definition:** The method of finding the number of trailing zeroes of $N!$ in prime base p is as follows:

$$\left\lfloor \frac{N}{p} \right\rfloor + \left\lfloor \frac{\left\lfloor \frac{N}{p} \right\rfloor}{p} \right\rfloor + \left\lfloor \frac{\left\lfloor \frac{\left\lfloor \frac{N}{p} \right\rfloor}{p} \right\rfloor}{p} \right\rfloor + \ldots.$$

Lemma 1: 3^a divides $(2+2a)!$ if and only if $2 + 2a = 3^m + 3^n$ for some integer m, n.

Proof: Note that

$$a \leq \sum_{i=1}^{\infty} \left\lfloor \frac{2a+2}{3^i} \right\rfloor < \sum_{i=1}^{\infty} \frac{2a+2}{3^i} = a + 1.$$

Thus,

$$\sum_{i=1}^{\infty} \left\{ \frac{2a+2}{3^i} \right\} = 1.$$

Write

$$2a+2=\sum_{k=0}^{\infty} d_k 3^k, d_k \in \{0,1,2\}$$

so

$$\sum_{i=1}^{\infty}\left\{\frac{\sum_{k=0}^{\infty} d_k 3^k}{3^i}\right\}=\sum_{i=1}^{\infty}\frac{\sum_{k=0}^{i-1} d_k 3^k}{3^i}=1$$

or

$$\sum_{k=0}^{\infty}\sum_{i=k+1}^{\infty}\frac{d_k 3^k}{3^i}=\sum_{k=0}^{\infty}\frac{d_k}{2}=1.$$

Thus, $2a+2=3^m+3^n$, as desired.

In base 2, $\left(2+2^{96}\right)!$, when written in base 2, has exactly

$$\left(1+2^{95}+2^{94}+2^{93}+\cdots+2^2+2^1+2^0\right)=2^{96}$$

trailing zeroes. Hence, when written in base $2^8=256$, there are exactly $\frac{2^{96}}{8}=2^{93}$ zeroes.

In base $p \geq 11$, there are:

$$\left\lfloor\frac{2+2^{96}}{p}\right\rfloor+\left\lfloor\frac{\left\lfloor\frac{2+2^{96}}{p}\right\rfloor}{p}\right\rfloor+\left\lfloor\frac{\left\lfloor\frac{\left\lfloor\frac{2+2^{96}}{p}\right\rfloor}{p}\right\rfloor}{p}\right\rfloor+\cdots \leq \left(2+2^{96}\right)\left(\frac{1}{11}+\frac{1}{11^2}+\frac{1}{11^3}\right)+\ldots=\left(2+2^{96}\right)\left(\frac{1}{10}\right)<2^{93}$$

trailing zeroes. Hence, the maximum possible prime divisor of the base is 7.

In base 7, there are

$$\left\lfloor\frac{2+2^{96}}{7}\right\rfloor+\left\lfloor\frac{\left\lfloor\frac{2+2^{96}}{7}\right\rfloor}{7}\right\rfloor+\left\lfloor\frac{\left\lfloor\frac{\left\lfloor\frac{2+2^{96}}{7}\right\rfloor}{7}\right\rfloor}{7}\right\rfloor+\cdots \leq \left(2+2^{96}\right)\left(\frac{1}{7}+\frac{1}{7^2}+\frac{1}{7^3}+\ldots\right) \leq \left(2+2^{96}\right)\left(\frac{1}{6}\right)<2^{94},$$

so the maximum prime power of 7 is 1. To prove that this is indeed the maximum, notice that

$$\left\lfloor\frac{2+2^{96}}{7}\right\rfloor+\left\lfloor\frac{\left\lfloor\frac{2+2^{96}}{7}\right\rfloor}{7}\right\rfloor+\left\lfloor\frac{\left\lfloor\frac{\left\lfloor\frac{2+2^{96}}{7}\right\rfloor}{7}\right\rfloor}{7}\right\rfloor+\cdots>\left\lfloor\frac{2+2^{96}}{7}\right\rfloor>\frac{2+2^{96}}{7}-1>\frac{2^{96}}{8}=2^{93}.$$

Similarly, the maximum prime power of 5 is 1. The prime power of 3 can be 3. To see that it cannot be 4, note that:

$$2+2^{96} \equiv 3 \pmod 9.$$

From the lemma, $2+2^{96}=3^a+3^b \equiv 3 \pmod 9$, so exactly one of a, b equals 3. But then $2^{96}-1=3^a$, but $2^3-1=7$ divides $2^{96}-1$, a contradiction. Hence, the maximum prime power of 3 is 3.

In base $B=p_1^{e_1}p_2^{e_2}\ldots p_k^{e_k}$, note that $\left(2+2^{96}\right)!$ has exactly

$$\min\left(\left\{\left\lfloor\frac{1}{e_i}\left\lfloor\frac{2+2^{96}}{p_i}\right\rfloor+\left\lfloor\frac{\left\lfloor\frac{2+2^{96}}{p_i}\right\rfloor}{p_i}\right\rfloor+\left\lfloor\frac{\left\lfloor\frac{\left\lfloor\frac{2+2^{96}}{p_i}\right\rfloor}{p_i}\right\rfloor}{p_i}\right\rfloor+\ldots\right\rfloor\right\}\right)$$

trailing zeroes. Since this must equal exactly 2^{93}, and the value for $p_i = 3, 5, 7$ cannot equal exactly 2^{93}, B must divide 2^8 exactly. We can then choose a nonnegative integer at most the maximum for each of the other prime bases.

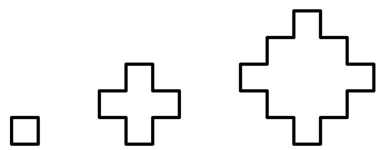
It follows that the answers are:

(a) $2^8 = \boxed{256}$

(b) $2^8 \cdot 3^3 \cdot 5^1 \cdot 7^1 = \boxed{241920}$

(c) $(3+1)(1+1)(1+1) = \boxed{16}$

8. By graphing out the given function in the problem, the function traces out a polygon composed of the union of all unit squares with centers of at most $n-1$ rectilinear distance from the origin. In particular, it creates a polygon with all sides of length 1, at right angles with each other and to the coordinate axes, as shown:



and so on.

(a) Taking the projection to either the x-axis or y-axis, we get a length of $2n-1$. Since all sides are orthogonal to the coordinate axes, every length contributes either 0 or 1 to this amount. Since we can take the projection to the left, right, top, or bottom, the perimeter is $(1)(4)(2n-1)+8n-4 \equiv 4 \pmod 8$.

(b) Since all sides are at either right angles or reflex right angles to each other, the sides alternate from parallel to perpendicular to the x-axis. Therefore, by marking every two units around, we select every other side, so we pick either all the parallel or all the perpendicular sides. Note that if we rotate the diagram by 90 degrees around the origin, we map the parallel sides to the perpendicular sides, so we may assume WLOG that point O lies on a segment parallel to the x-axis. Above the line $y = 0$, it forms an isosceles right triangle with height of $n-1$ to the hypotenuse; similarly, below the line $y = 0$ it forms a right triangle with height of $n-1$. Joining the two forms a parallelogram with a constant base and constant height, so the area $[S_{nO}]$ is invariant.

(c) We compute the area of R_n first: a straightforward computation yields $[R_n] = 2(n-1)^2+2(n-1) = 2n^2-2n$. Additionally, $[S_{nO}] = n^2+(n-1)^2 = 2n^2-2n+1$, so $[R_n]+[S_{nO}] = 4n^2-4n+1 = (2n-1)^2$, as claimed.