NIMO 2011

1. This is an eight-problem exam. Your team of up to four will be allotted 60 minutes to complete the exam.
2. You must show work and justify your reasoning to receive credit. Answers without justification will receive little to no credit. Conversely, a well-written solution with a few minor errors may receive near-full or full credit.
3. No aids other than scratch paper, graph paper, rulers, compasses, and protractors are permitted. In particular, calculators, slide rules, or other computational aids are not allowed. Use of computers is permitted only for communication among team members and for submitting solutions.
4. All solutions must be handwritten on the provided answer sheets. Please make sure your handwriting is legible and dark enough to be processed by a machine - we cannot grade solutions that are unreadable. No page should include solutions to more than one problem.
5. You must submit your solutions on or before January 15, 2011. You must submit your solutions by one of the following methods:
   (a) Mail all of your solutions with cover sheet in order in a single envelope addressed to:

       NIMO
       P.O. Box 988
       Pleasanton, CA 94566

       Solutions must be postmarked on the day after the test was downloaded (or earlier).

   (b) Upload your solutions in a single PDF file through your team's account at our site. You will be allotted up to 5 minutes after the contest to scan and upload your solutions.
6. Your team's results will be posted on the NIMO website when available. Your team captain will also be sent an email when scores are available.
7. **Please read all of the rules on the NIMO website before beginning the exam.**

The Problems

1. A point (x, y) in the first quadrant lies on a line with intercepts $(a, 0)$ and $(0, b)$, with $a, b > 0$. Rectangle M has vertices $(0, 0)$, $(x, 0)$, (x, y), and $(0, y)$, while rectangle N has vertices (x, y), (x, b), (a, b), and (a, y). What is the ratio of the area of M to that of N?

2. Two sequences $\{a_i\}$ and $\{b_i\}$ are defined as follows: $\{a_i\} = 0, 3, 8, \ldots, n^2 - 1, \ldots$ and $\{b_i\} = 2, 5, 10, \ldots, n^2+1, \ldots$. If both sequences are defined with i ranging across the natural numbers, how many numbers belong to both sequences?

3. Billy and Bobby are located at points A and B, respectively. They each walk directly toward the other point at a constant rate; once the opposite point is reached, they immediately turn around and walk back at the same rate. The first time they meet, they are located 3 units from point A; the second time they meet, they are located 10 units from point B. Find all possible values for the distance between A and B.

4. In the following alpha-numeric puzzle, each letter represents a different non-zero digit. What are all possible values for $b + e + h$?

	a	b	c
	d	e	f
+	g	h	i
1	6	6	5

5. We have eight light bulbs, placed on the eight lattice points in space that are $\sqrt{3}$ units away from the origin. Each light bulb can either be turned on or off. These lightbulbs are unstable, however. If two light bulbs that are at most 2 units apart are both on simultaneously, they both explode. Given that no explosions take place, how many possible configurations of on/off light bulbs exist?

6. Circle $\odot O$ with diameter $\overline{AB}$ has chord $\overline{CD}$ drawn such that $\overline{AB}$ is perpendicular to $\overline{CD}$ at P. Another circle $\odot A$ is drawn, sharing chord $\overline{CD}$. A point Q on minor arc $\overline{CD}$ of $\odot A$ is chosen so that $\measuredangle AQP + \measuredangle QPB = 60°$. Line l is tangent to $\odot A$ through Q and a point X on l is chosen such that $PX = BX$. If $PQ = 13$ and $BQ = 35$, find QX.

7. The number $\left(2 + 2^{96}\right)!$ has 2^{93} trailing zeroes when expressed in base B.
   (a) Find the minimum possible B.
   (b) Find the maximum possible B.
   (c) Find the total number of possible B.

8. Define $f(x)$ to be the nearest integer to x, with the greater integer chosen if two integers are tied for being the nearest. For example, $f(2.3) = 2$, $f(2.5) = 3$, and $f(2.7) = 3$. Define $[A]$ to be the area of region A. Define region R_n, for each positive integer n, to be the region on the Cartesian plane which satisfies the inequality $f(|x|) + f(|y|) < n$. We pick an arbitrary point O on the perimeter of R_n, and mark every two units around the perimeter with another point. Region S_{nO} is defined by connecting these points in order.
   (a) Prove that the perimeter of R_n is always congruent to 4 (mod 8).
   (b) Prove that $[S_{nO}]$ is constant for any O.
   (c) Prove that $[R_n] + [S_{nO}] = (2n - 1)^2$.